Letter

Inertial forces in the Maxey–Riley equation in nonuniform flows

Bhargav Rallabandi [*]
Department of Mechanical Engineering, University of California, Riverside, California 92521, USA

(Received 31 October 2020; accepted 22 December 2020; published 7 January 2021)

The Maxey–Riley equation describes the motion of a spherical particle suspended in a spatially nonuniform, time-dependent flow, and finds applications in a wide range of flow situations. We reexamine the hydrodynamics underlying the Maxey–Riley equation to find additional inertial forces associated with second gradients of the background flow velocity, not accounted for in the original framework. These forces amplify inertial Faxén terms threefold, while also contributing advective terms that are quadratic in fluid velocity and may exceed Faxén forces in some flows. We present a more comprehensive form of the Maxey–Riley equation that includes these contributions, and discuss its implications for particle dynamics in flows with curvature.

DOI: 10.1103/PhysRevFluids.6.L012302

The Maxey–Riley equation [1,2] is a statement of momentum conservation of a spherical particle suspended in a spatially nonuniform, time-dependent fluid flow. Along with its variants (cf. the review by Michaelidis [3]) it has been widely used to model particle transport in a range of applications including microfluidic particle sorting [4,5], multiphase turbulence [6], and ocean surface transport [7]. Maxey and Riley [1] consider a rigid sphere of radius a and mass m_p suspended in a Newtonian fluid of density ρ and viscosity μ. The flow in the absence of the particle (referred to as the background flow) has a known velocity $\mathbf{u}(\mathbf{x}, t)$ that satisfies the incompressible Navier–Stokes equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma}^{(0)} + \rho\mathbf{g}, \quad \nabla\cdot\mathbf{u} = 0. \tag{1}$$

Here, $\boldsymbol{\sigma}^{(0)}$ is the stress tensor of the background flow and $\mathbf{g}$ is the acceleration due to gravity. The particle's center is located at $\mathbf{X}(t)$ and translates with velocity $\mathbf{V}(t) = d\mathbf{X}/dt$, producing a disturbance flow $\mathbf{u}'(\mathbf{x}, t)$ (stress $\boldsymbol{\sigma}'$) so that the net flow $\mathbf{v} = \mathbf{u} + \mathbf{u}'$ (stress $\boldsymbol{\sigma}^{(0)} + \boldsymbol{\sigma}'$) continues to solve the incompressible Navier–Stokes equations.

The fluid exerts a force $\int_{S_p} \mathbf{n}\cdot(\boldsymbol{\sigma}^{(0)} + \boldsymbol{\sigma}')dS$ on the particle, where $\mathbf{n}$ is the fluid-facing unit normal to the particle surface S_p. The force contributed by the background flow, which we will return to as the focus of this Letter, is defined by $\mathbf{F}^{(0)} = \int_{S_p} \mathbf{n}\cdot\boldsymbol{\sigma}^{(0)}dS$ and is approximated by Maxey and Riley as (cf. (25) of [1])

$$\mathbf{F}^{(0)} \approx -m_f\mathbf{g} + m_f\frac{D\mathbf{u}}{Dt}, \tag{2}$$

where $m_f = \frac{4}{3}\pi a^3\rho$ and $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)$ is the material derivative in the background flow. Maxey and Riley then use the unsteady Stokes equations to approximate the disturbance flow $\mathbf{u}'$ and calculate the associated force $\mathbf{F}' = \int_{S_p} \mathbf{n}\cdot\boldsymbol{\sigma}'dS$. Applying Newton's second law to the particle

[*]bhargav@engr.ucr.edu

($m_p d\mathbf{V}/dt = m_p\mathbf{g} + \mathbf{F}^{(0)} + \mathbf{F}'$) yields the Maxey–Riley equation which, in the form originally derived [1] and remains widely used [8–11], reads

$$m_p \frac{d\mathbf{V}}{dt} = (m_p - m_f)\mathbf{g} + m_f \frac{D\mathbf{u}}{Dt}\bigg|_{\mathbf{X}(t)} - \frac{1}{2} m_f \frac{d}{dt}\left(\mathbf{V} - \mathbf{u} - \frac{a^2}{10}\nabla^2\mathbf{u}\right)\bigg|_{\mathbf{X}(t)}$$
$$- 6\pi\mu a\left(\mathbf{V} - \mathbf{u} - \frac{a^2}{6}\nabla^2\mathbf{u}\right)\bigg|_{\mathbf{X}(t)}$$
$$- 6\pi\mu a^2 \int_0^t \frac{d\tau}{\sqrt{\pi\nu(t-\tau)}} \frac{d}{d\tau}\left(\mathbf{V}(\tau) - \mathbf{u} - \frac{a^2}{6}\nabla^2\mathbf{u}\right)\bigg|_{\mathbf{X}(\tau)}, \tag{3}$$

where $\nu = \mu/\rho$. Equation (3) accounts for buoyancy, fluid inertia, added-mass, viscous drag, and the Basset–Boussinseq–Oseen history force. The latter three forces also contain Faxén terms (involving $a^2\nabla^2\mathbf{u}$) whose importance is determined by the particle size relative to the characteristic length scale of the flow and are relevant across a range of flows involving either particles or bubbles [4,12–14].

Several modifications of (3) have also been developed, often to account for the effects of advective inertia on $\mathbf{F}'$ [3]. The most widely used of these [8,15,16] is due to Auton *et al.* [17] and replaces the added mass term $\frac{1}{2} m_f \frac{d}{dt}(\mathbf{u} + \frac{a^2}{10}\nabla^2\mathbf{u})$ by $\frac{1}{2} m_f \frac{D}{Dt}(\mathbf{u} + \frac{a^2}{10}\nabla^2\mathbf{u})$. Lovalenti and Brady [18] showed that advective inertia suppresses history forces. Other authors have modified (3) to include forces from shear-induced lift [12] (see [19]) or interactions with boundaries [5,20].

The focus of this work is the background force $\mathbf{F}^{(0)}$, independent of effects associated with the disturbance flow. We show that (2) is merely the leading term of an expansion that includes additional terms involving the velocity curvature tensor $\nabla\nabla\mathbf{u}$. Since the background flow is defined everywhere (in particular inside the volume V_p occupied by the particle), we use the divergence theorem to transform the surface integral $\mathbf{F}^{(0)} = \int_{S_p} \mathbf{n}\cdot\boldsymbol{\sigma}^{(0)} dS$ into an integral over the particle volume and invoke (1) to write

$$\mathbf{F}^{(0)} = \int_{V_p} \nabla\cdot\boldsymbol{\sigma}^{(0)} dV = \rho \int_{V_p} \left(\frac{D\mathbf{u}}{Dt} - \mathbf{g}\right) dV. \tag{4}$$

Approximating $D\mathbf{u}/Dt$ as uniform over the particle volume leads to the Maxey–Riley result (2) (cf. Eqs. (20)–(25) of [1]). More generally $D\mathbf{u}/Dt$ has spatial variations and can be expressed inside V_p using a Taylor series about the instantaneous particle center as

$$\frac{D\mathbf{u}}{Dt} = \frac{D\mathbf{u}}{Dt}\bigg|_{\mathbf{X}(t)} + \mathbf{r}\cdot\nabla\frac{D\mathbf{u}}{Dt}\bigg|_{\mathbf{X}(t)} + \frac{1}{2}\mathbf{rr}:\nabla\nabla\left(\frac{D\mathbf{u}}{Dt}\right)\bigg|_{\mathbf{X}(t)} + \cdots, \tag{5}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{X}(t) = r\mathbf{n}$ is the instantaneous position vector relative to the center of the particle. Writing $\int_{V_p}(\cdot)dV = \int_0^a \int_\Omega (\cdot)\, r^2 d\Omega dr$, where Ω is the solid angle, and evaluating the integral in (4) yields

$$\mathbf{F}^{(0)} = -m_f\mathbf{g} + m_f\left\{\frac{D\mathbf{u}}{Dt} + \frac{a^2}{10}\nabla^2\left(\frac{D\mathbf{u}}{Dt}\right) + \cdots\right\}\bigg|_{\mathbf{X}(t)}, \tag{6}$$

up to a leading error term of $m_f \frac{a^4}{280}\nabla^4\left(\frac{D\mathbf{u}}{Dt}\right)\big|_{\mathbf{X}(t)}$. Contrasting (6) with (2) makes clear the additional contributions in (6) involving the curvature of the background velocity $\mathbf{u}$. Note that the original derivation of Maxey and Riley uses expansions analogous to (5) to calculate $\mathbf{F}'$ [leading to Faxén terms in (3)] but neglects their contributions to $\mathbf{F}^{(0)}$ [1]. The additional term in (6) was previously recognized by Gatignol (cf. Eq. (44) of [2]) and by others [13,21,22] in different contexts, although their relevance to the Maxey–Riley framework appears not to have been noted.

Using (6) in the particle momentum equation $m_p d\mathbf{V}/dt = m_p\mathbf{g} + \mathbf{F}^{(0)} + \mathbf{F}'$ while retaining the unsteady Stokes approximation for $\mathbf{F}'$ as in [1] yields our main result for the dynamics of the particle,

$$m_p\frac{d\mathbf{V}}{dt} = (m_p - m_f)\mathbf{g} + m_f\left\{\frac{D}{Dt}\left(\mathbf{u} + \frac{a^2}{10}\nabla^2\mathbf{u}\right) + \frac{a^2}{10}\nabla^2\mathbf{u}\cdot\nabla\mathbf{u} + \frac{a^2}{5}\nabla\mathbf{u} : \nabla\nabla\mathbf{u}\right\}\Big|_{\mathbf{X}(t)}$$
$$- \frac{1}{2}m_f\frac{d}{dt}\left(\mathbf{V} - \mathbf{u} - \frac{a^2}{10}\nabla^2\mathbf{u}\right)\Big|_{\mathbf{X}(t)} - 6\pi\mu a\left(\mathbf{V} - \mathbf{u} - \frac{a^2}{6}\nabla^2\mathbf{u}\right)\Big|_{\mathbf{X}(t)}$$
$$- 6\pi\mu a^2\int_0^t \frac{d\tau}{\sqrt{\pi\nu(t-\tau)}}\frac{d}{d\tau}\left(\mathbf{V}(\tau) - \mathbf{u} - \frac{a^2}{6}\nabla^2\mathbf{u}\right)\Big|_{\mathbf{X}(\tau)}, \tag{7}$$

where we have written out $\nabla^2(\frac{D\mathbf{u}}{Dt}) = \frac{D(\nabla^2\mathbf{u})}{Dt} + \nabla^2\mathbf{u}\cdot\nabla\mathbf{u} + 2\nabla\mathbf{u} : \nabla\nabla\mathbf{u}$. Equation (7) is a self-consistent modification of (3) to include contributions from the curvature of the background flow, and should be used in situations where gradients of the flow are appreciable on the scale of the particle. Due to the approximation of $\boldsymbol{F}'$, (7) is formally accurate to linear order in $\boldsymbol{u}$ for locally quadratic flows, even though $\boldsymbol{F}^{(0)}$ fully accounts for the inertia of the background flow.

The primary quantitative effect of including the background curvature terms in (7) is the modification of Faxén accelerations from $\frac{1}{20}m_f a^2\frac{d}{dt}(\nabla^2\mathbf{u})$ to $\frac{1}{20}m_f a^2(\frac{d}{dt} + 2\frac{D}{Dt})\nabla^2\mathbf{u}$, which is a roughly threefold increase (typically $d/dt \approx D/Dt$ for small particles). An important qualitative addition in (7) is the term $\frac{1}{5}m_f a^2\nabla\mathbf{u} : \nabla\nabla\mathbf{u}$, which is the only force that depends on the full velocity curvature tensor $\nabla\nabla\mathbf{u}$, and may dominate Faxén terms in flows with small $\nabla^2\mathbf{u}$. For example, a particle that is density matched with the fluid and suspended in background potential flow ($\nabla^2\mathbf{u} = 0$) deviates from fluid trajectories due to the $m_f a^2\nabla\mathbf{u} : \nabla\nabla\mathbf{u}$ term even though all Faxén forces (which scale with $a^2\nabla^2\mathbf{u}$) are identically zero.

We emphasize that the curvature contributions discussed here are part of the force from the background flow $\mathbf{F}^{(0)}$ and have no bearing on forces due to the disturbance flow $\mathbf{F}'$. In particular, contributions to $\mathbf{F}'$ associated with the advective inertia of $\mathbf{u}'$ have been neglected in (7) as in the original Maxey–Riley formulation. These terms can be important and may be separately accounted for as discussed earlier, e.g., the Auton added mass or shear-induced lift. Another inertial contribution to $\mathbf{F}'$ involving flow curvature can be gleaned from the analysis of Lhuillier [23], who showed that for potential flow (both background and disturbance) the disturbance flow contributes a force of $\frac{2}{15}m_f a^2\nabla\mathbf{u} : \nabla\nabla\mathbf{u}$. This term may be added to the right side of (7), although it is unclear how it is modified in a general setting with finite vorticity.

In summary, the present work presents a modification to the Maxey–Riley equation due to the inertia of a spatially varying background flow, leading to the main result (7). Relative to the original Maxey–Riley formalism, the additional force contributions amplify inertial Faxén terms by a factor of 3, while also introducing terms that are quadratic in the background flow and depend on the velocity gradient and curvature. These modifications are significant when the size of the particle is appreciable relative to characteristic length scale of the flow, as is the case in many microscopic and macroscopic systems [4,12,13]. Their incorporation into the Maxey–Riley equation may further broaden its scope as a tool for inertial particle dynamics in nonuniform flows.

The author thanks Siddhansh Agarwal, Sascha Hilgenfeldt, and Howard A. Stone for fruitful discussions. The author is grateful for support from a Hellman Fellowship and the Bourns College of Engineering at UCR.

[1] M. R. Maxey and J. J. Riley, Equation of motion for a small rigid sphere in a nonuniform flow, Phys. Fluids **26**, 883 (1983).

[2] R. Gatignol, The Faxén formulae for a rigid particle in an unsteady non-uniform Stokes flow, Journal de Mécanique théorique et appliquée **1**, 143 (1983).
[3] E. E. Michaelides, The transient equation of motion for particles, bubbles, and droplets, J. Fluids. Eng. **119**, 233 (1997).
[4] H. Haddadi, H. Naghsh-Nilchi, and D. Di Carlo, Separation of cancer cells using vortical microfluidic flows, Biomicrofluidics **12**, 014112 (2018).
[5] R. Thameem, B. Rallabandi, and S. Hilgenfeldt, Fast inertial particle manipulation in oscillating flows, Phys. Rev. Fluids **2**, 052001 (2017).
[6] V. Mathai, D. Lohse, and C. Sun, Bubbly and buoyant particle–laden turbulent flows, Annu. Rev. Condens. Matter Phys. **11**, 529 (2020).
[7] F. J. Beron-Vera, M. J. Olascoaga, and P. Miron, Building a Maxey–Riley framework for surface ocean inertial particle dynamics, Phys. Fluids **31**, 096602 (2019).
[8] M. R. Maxey, The equation of motion for a small rigid sphere in a nonuniform or unsteady flow, in *Gas-Solid Flows* (The American Society of Mechanical Engineers, New York, 1993), Vol. 166, pp. 57–62.
[9] M. H. Kobayashi and C. F. Coimbra, On the stability of the Maxey–Riley equation in nonuniform linear flows, Phys. Fluids **17**, 113301 (2005).
[10] E. Mograbi and E. Bar-Ziv, On the asymptotic solution of the Maxey–Riley equation, Phys. Fluids **18**, 051704 (2006).
[11] M. Sudharsan, S. L. Brunton, and J. J. Riley, Lagrangian coherent structures and inertial particle dynamics, Phys. Rev. E **93**, 033108 (2016).
[12] K. Chong, S. D. Kelly, S. Smith, and J. D. Eldredge, Inertial particle trapping in viscous streaming, Phys. Fluids **25**, 033602 (2013).
[13] E. Calzavarini, R. Volk, M. Bourgoin, E. Lévêque, J.-F. Pinton, and F. Toschi, Acceleration statistics of finite-sized particles in turbulent flow: The role of Faxén forces, J. Fluid. Mech. **630**, 179 (2009).
[14] V. Mathai, V. N. Prakash, J. Brons, C. Sun, and D. Lohse, Wake-Driven Dynamics of Finite-Sized Buoyant Spheres in Turbulence, Phys. Rev. Lett. **115**, 124501 (2015).
[15] G. Haller and T. Sapsis, Where do inertial particles go in fluid flows? Physica D **237**, 573 (2008).
[16] M. Farazmand and G. Haller, The Maxey–Riley equation: Existence, uniqueness and regularity of solutions, Nonlinear Anal. Real World Appl. **22**, 98 (2015).
[17] T. Auton, J. Hunt, and M. Prud'Homme, The force exerted on a body in inviscid unsteady non-uniform rotational flow, J. Fluid. Mech. **197**, 241 (1988).
[18] P. M. Lovalenti and J. F. Brady, The hydrodynamic force on a rigid particle undergoing arbitrary time-dependent motion at small Reynolds number, J. Fluid. Mech. **256**, 561 (1993).
[19] P. G. T. Saffman, The lift on a small sphere in a slow shear flow, J. Fluid Mech. **22**, 385 (1965).
[20] S. Agarwal, B. Rallabandi, and S. Hilgenfeldt, Inertial forces for particle manipulation near oscillating interfaces, Phys. Rev. Fluids **3**, 104201 (2018).
[21] P. M. Lovalenti, J. F. Brady, and H. A. Stone, The reciprocal theorem and expressions for the particle force. Appendix A to Lovalenti & Brady, J. Fluid. Mech. **256**, 591 (1993).
[22] H. A. Stone, J. F. Brady, and P. M. Lovalenti, Inertial effects on the rheology of suspensions and on the motion of individual particles (2016), preprint.
[23] D. Lhuillier, On the equation of motion of a rigid sphere in a non uniform and accelerated inviscid fluid: Incidence on two-phase flow equation, Mech. Res. Commun. **9**, 295 (1982).