Hydrodynamic force on a sphere normal to an obstacle due to a non-uniform flow

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For a small sphere suspended in a background fluid flow near an obstacle, we calculate the hydrodynamic force on the sphere in the direction normal to the boundary of the obstacle. Using the Lorentz reciprocal theorem, we obtain analytical expressions for the normal force in the Stokes flow limit, valid for arbitrary separations of the particle from the obstacle, both for solid obstacles and those with free surfaces. The main effect of the boundary is to produce a normal force proportional to extensional flow gradients in the vicinity of the particle. The strength of this force is greatest when the separation between the surfaces of the particle and the obstacle is small relative to the particle size. While the magnitude of the force weakens for large separations between the sphere and the obstacle (decaying quadratically with separation distance), it can significantly modify Faxén's law even at modestly large separation distances. In addition, we find a second force contribution due to the curvature of the background flow normal to the obstacle, which is also important when the sphere is close to the obstacle. The results of the theory are of importance to the dynamics of particles in confined geometries, whether bounded by a solid obstacle, the wall of a channel or a gas bubble.

Key words: low-Reynolds-number flows, particle/fluid flow, suspensions

1. Introduction

The motion of suspended particles in viscous fluids is a much studied research topic and finds a wide range of applications in industrial and natural settings, as well as in the study of fluid flows. A number of industrial processes involve the transport of particles over long distances over which segregation of particles (and therefore possible destabilization of a suspension) can be a practical concern. By contrast, it may be desirable to engineer devices that induce controlled particle segregation based on a physical property of the particle such as its size or shape, e.g. field flow fractionation (Giddings, Yang & Myers 1976). In situations where particles faithfully follow fluid trajectories, i.e. when they behave as passive tracers, their motion is commonly used to characterize and visualize the flow itself.

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An accurate quantification of the hydrodynamic resistance to the motion of a suspended particle is key to describing its motion. The classical Stokes law for the drag on a particle was modified by Faxén to include the effect of an externally imposed pressure gradient and by Oseen to account for finite fluid inertia (Happel & Brenner 1965; Leal 1980). Saffman (1965) showed that a particle experiences a lift force in a shear flow at small but finite Reynolds numbers due to Oseen corrections, and various generalizations have been obtained subsequently (McLaughlin 1991; Asmolov 1995). The effect of an unsteady background flow on the motion of a sphere in an unbounded fluid has also been studied (Lovalenti & Brady 1993, 1995; Asmolov & McLaughlin 1999).

Many flows with particles, such as the flow of a suspension through a pipe or a microfluidic channel, or the flow of particles around an obstacle, involve motion near a solid boundary. The presence of the boundary modifies the hydrodynamics of particles suspended in the flow, in particular at low Reynolds numbers where hydrodynamic interactions are long range. A number of studies have quantified the dynamics of a single sphere near a wall in the Stokes flow limit (Brenner 1961; Goldman, Cox & Brenner 1967*a*,*b*), and several others have quantified the effect of inertia in such configurations (Segré & Silberberg 1962; Ho & Leal 1974; Vasseur & Cox 1977; McLaughlin 1993; Hogg 1994; Hood, Lee & Roper 2015). Much research has focused on the hydrodynamics of dilute suspensions in which boundary effects are weak (Batchelor & Green 1972*a*; Hinch 1977; Haber & Brenner 1999; Bhattacharya, Błlawzdziewicz & Wajnryb 2006; Zurita-Gotor, Błlawzdziewicz & Wajnryb 2007; Feuillebois *et al.* 2015) and on the dynamics of pairs of spheres in linear flows (Goldman, Cox & Brenner 1966; Batchelor & Green 1972*b*; Nir & Acrivos 1973; Jeffrey & Onishi 1984; Swan & Brady 2007). Other studies have considered close-range particle–particle and particle–wall interactions (Goren & O'Neill 1971; Sangani, Acrivos & Peyla 2011; Cardinaels & Stone 2015).

Several researchers have, to good effect, combined far-field descriptions of particle hydrodynamics with near-field corrections for lubrication when particles are close to each other or to other boundaries in the flow (Durlofsky, Brady & Bossis 1987; Brady & Bossis 1988; Bossis, Meunier & Sherwood 1991; Swan & Brady 2011; Aponte-Rivera & Zia 2016; Gallier *et al.* 2016). Another approach is to simulate the flow numerically accounting for the presence of suspended particles, thereby computing both the flow and the particle trajectory simultaneously (Dance & Maxey 2003; Kempe & Fröhlich 2012*b*; Haddadi & Morris 2015). Such methods, however, also typically require additional corrections for lubrication when separation distances between solid surfaces are small (Ladd & Verberg 2001; Nguyen & Ladd 2002; Kempe & Fröhlich 2012*a*; Izard, Bonometti & Lacaze 2014).

In this paper, we develop a general theory of the force on a rigid spherical particle suspended in a general quadratic background flow in the vicinity of a large obstacle; we determine the force on the particle in a direction normal to the surface of the obstacle. Working within the Stokes flow limit, we use the Lorentz reciprocal theorem along with a local expansion of the background flow about the particle centre to obtain explicit expressions for the normal force as a function of the particle motion and the local properties of the background flow. The theory provides exact results for arbitrary distances between the surface of the particle and the obstacle, whose boundary is allowed to be either rigid or a free surface. To aid the analysis, we focus on situations where the obstacle's surface is itself spherical, although some general conclusions can be drawn for non-spherical obstacle boundaries, as discussed in subsequent sections.
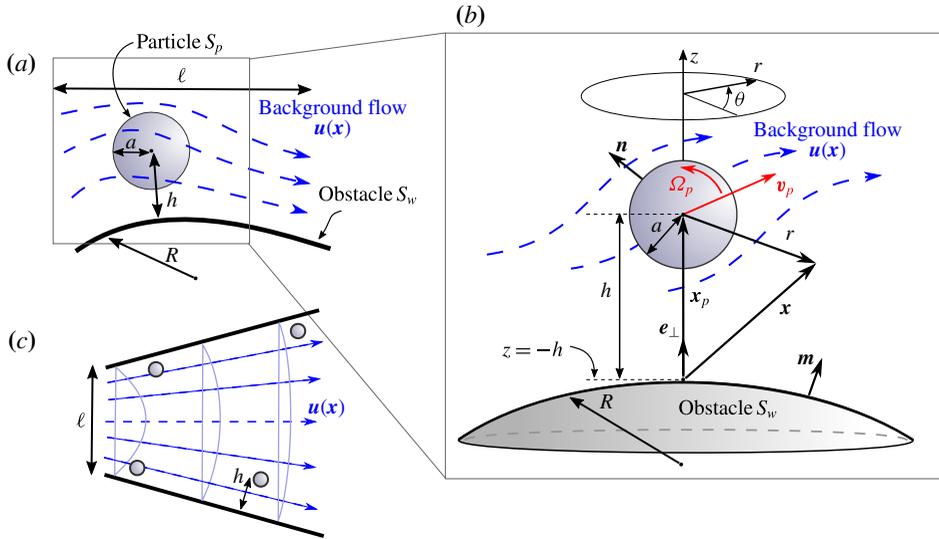
FIGURE 1. (Colour online) (*a*) General set-up of the problem of a small spherical particle of radius a immersed in a background flow field $\boldsymbol{u}(\boldsymbol{x})$ at a distance h from a large obstacle of radius of curvature R. The characteristic scale of the flow is $\ell \gg a$; $\ell = O(R)$ in many situations. (*b*) A sketch of the geometry relevant to the disturbance flow problem in the vicinity of the particle. The obstacle surface is assumed to be locally spherical with a radius $R \gtrsim \ell \gg a$. (*c*) An example with $\ell \ll R$ is the pressure-driven flow of particle-laden fluids through channels of non-uniform shape; as sketched, the walls are locally planar.

Employing a quadratic expansion of the background flow about the spherical particle's centre (see figure 1), we obtain a general expression for the normal component of the force of the form

$$F_{\perp} = \boldsymbol{F} \cdot \boldsymbol{e}_{\perp} = 6\pi\mu a \left[\left\{ -\mathcal{A}\,(\boldsymbol{v}_p - \boldsymbol{u}) - a\,\mathcal{B}\,(\boldsymbol{e}_{\perp} \cdot \boldsymbol{\nabla u}) \right. \right.$$
$$\left. \left. + \frac{a^2}{2}\mathcal{C}\,(\boldsymbol{e}_{\perp}\boldsymbol{e}_{\perp} : \boldsymbol{\nabla\nabla u}) + \frac{a^2}{2}\mathcal{D}\,\nabla^2\boldsymbol{u} \right\} \cdot \boldsymbol{e}_{\perp} \right]_{\boldsymbol{x}_p}, \quad (1.1)$$

where $\boldsymbol{e}_{\perp}$ is the unit normal to the obstacle pointing towards the particle centre, a is the particle's radius and μ is the viscosity of the fluid. Also, $\boldsymbol{v}_p$ and $\boldsymbol{u}$ are, respectively, the particle and ambient flow velocity and $\boldsymbol{x}_p$ represents the position of the particle's centre; cf. figure 1. The scalar quantities $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are dimensionless hydrodynamic resistances that depend on (i) the separation distance between the particle and the obstacle and (ii) the size of the particle relative to that of the obstacle; these resistances are calculated explicitly in this paper. We recover Faxén's law and the results of lubrication theory, respectively, for large and small separations between the sphere and the boundary. The results of the theory can be used to compute approximate trajectories of particles near boundaries consistently without the need to include additional lubrication terms to prevent surface-to-surface contact.

The paper is organized as follows. Section 2 introduces the mathematical problem and briefly presents the scaling of the hydrodynamic force with particle size and the flow properties. In §3, we obtain a general expression for the normal force on the

particle by using the Lorentz reciprocal theorem for Stokes flow. We show that the normal force, up to cubic order in particle size, contains four different contributions involving moments of the background flow in the vicinity of the particle. In § 4, we obtain analytic expressions for these force contributions as functions of the distance of the particle from the boundary. In § 5, we discuss the asymptotic behaviour of the forces for large and small separations between the sphere and the boundary, showing that extensional contributions to the normal force are important even for modestly large separations between the particle and the obstacle. We also discuss the implications of the theory for the trapping of particles near surfaces and the motion of a force-free particle.

2. Problem set-up and general theory

We consider the flow of a viscous fluid (density ρ and viscosity μ) in the vicinity of a large obstacle of characteristic radius of curvature R, which is described by a velocity field $\boldsymbol{u}(\boldsymbol{x})$ that is assumed to be fully known. The flow has a characteristic velocity scale u_0 and a characteristic length scale ℓ that, in general, may or may not be related to R, cf. figure 1. As an example, gradients of flow past a large curved obstacle depend on the size of the obstacle (i.e. $\ell \sim R$). By contrast, in pressure-driven flow through a channel with plane walls ($R \rightarrow \infty$), flow gradients are determined by the width of the channel ($\ell \ll R$). Similarly, hydrodynamic interactions in dilute suspensions are associated with flow gradients on the scale of the interparticle separation ℓ and are generally unrelated to the curvature of nearby walls (Swan & Brady 2007).

At small Reynolds numbers ($Re_u \equiv \rho u_0 \ell / \mu \ll 1$), the flow is described by the Stokes equations

$$\nabla \cdot \boldsymbol{\sigma}_u = \boldsymbol{0} \quad \text{and} \quad \nabla \cdot \boldsymbol{u} = 0, \qquad (2.1a,b)$$

where $\boldsymbol{\sigma}_u = -p_u \boldsymbol{I} + \mu(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^{\mathrm{T}})$ is the stress tensor and p_u is the pressure. Although the precise shape of the obstacle is not ultimately important for the formalism developed here, we consider two classes of obstacles widely encountered in physical systems. The first corresponds to solid obstacles on whose surface the flow must satisfy the no-slip boundary condition (denoted BC I), and the second corresponds to obstacles with free surfaces, e.g. a clean gas bubble, on which tangential components of stress vanish (denoted BC II). The boundary conditions satisfied by the flow $(\boldsymbol{u}, \boldsymbol{\sigma}_u)$ on the surface S_w of the obstacle are accordingly either

$$\left.\begin{array}{rl} \text{BC I (no-slip):} & \boldsymbol{u} = \boldsymbol{0}, \quad \text{or} \\ \text{BC II (no-stress):} & \boldsymbol{u} \cdot \boldsymbol{m} = 0, \quad \text{and} \quad (\boldsymbol{m} \cdot \boldsymbol{\sigma}_u) \times \boldsymbol{m} = \boldsymbol{0} \end{array}\right\} \quad \text{on } S_w, \qquad (2.2)$$

where $\boldsymbol{m}$ denotes the unit normal to the surface of the obstacle, directed into the fluid (figure 1). The flow $(\boldsymbol{u}, \boldsymbol{\sigma}_u)$, which we refer to as the 'background flow', may satisfy some generally inhomogeneous boundary conditions at infinity.

We now introduce to the flow a rigid spherical particle of radius a with a centre at a distance h from the surface of the obstacle. The particle, whose centre is at a position $\boldsymbol{x}_p(t)$ translates with velocity $\boldsymbol{v}_p$ and rotates with angular velocity $\boldsymbol{\Omega}_p$, as indicated in figure 1. Due to the motion of the particle, the velocity field is no longer $\boldsymbol{u}(\boldsymbol{x})$, but rather

$$\boldsymbol{U}(\boldsymbol{x}) \equiv \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{v}(\boldsymbol{x}), \qquad (2.3)$$

where $\boldsymbol{v}(\boldsymbol{x})$ is the disturbance flow due to the moving particle. Assuming that the disturbance velocity scale v_0 is at most of the order of u_0, the net flow $\boldsymbol{U}$ also satisfies

the Stokes equations, the boundary conditions at the wall (either BC I or BC II as appropriate) and the conditions at infinity satisfied by $\boldsymbol{u}$. In addition, to observe the no-slip condition on the surface S_p of the rigid particle, $\boldsymbol{U}$ must satisfy

$$\boldsymbol{U} = \boldsymbol{v}_p + \boldsymbol{\Omega}_p \times \boldsymbol{r} \quad \text{on } S_p, \tag{2.4}$$

where $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{x}_p$ is the position of a point in space relative to the particle centre.

The disturbance flow $\boldsymbol{v}$ is therefore governed by

$$\nabla \cdot \boldsymbol{\sigma}_v = \boldsymbol{0} \quad \text{and} \quad \nabla \cdot \boldsymbol{v} = 0, \tag{2.5a,b}$$

$$\boldsymbol{v} = \boldsymbol{w}(\boldsymbol{x}) \equiv \boldsymbol{v}_p - \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{\Omega}_p \times \boldsymbol{r} \quad \text{for } \boldsymbol{x} \in S_p, \tag{2.5c}$$

$$\boldsymbol{v} \to \boldsymbol{0} \quad \text{as } |\boldsymbol{r}| \to \infty \quad \text{and either} \tag{2.5d}$$

$$\left. \begin{array}{ll} \text{BC I:} & \boldsymbol{v} = \boldsymbol{0}, \quad \text{or} \\ \text{BC II:} & \boldsymbol{v} \cdot \boldsymbol{m} = 0, \quad (\boldsymbol{m} \cdot \boldsymbol{\sigma}_u) \times \boldsymbol{m} = \boldsymbol{0} \end{array} \right\} \quad \text{on } S_w. \tag{2.5e}$$

Here, $\boldsymbol{\sigma}_v = -p_v \boldsymbol{I} + \mu(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}})$ is the stress tensor associated with the disturbance flow $\boldsymbol{v}$ (with p_v being the pressure). Also, $\boldsymbol{w}(\boldsymbol{x})$ is the disturbance flow surface velocity distribution on the sphere required for the net flow $\boldsymbol{U}$ to satisfy the no-slip condition on the particle.

Since disturbance flow quantities depend on the local background flow properties through (2.5c), $\boldsymbol{v}(\boldsymbol{x})$ is generally also time dependent along the trajectory of the particle. Inertial effects associated with this time dependence are small for $Re_u \ll 1$ and are neglected here, making the disturbance flow $\boldsymbol{v}(\boldsymbol{x})$ quasi-steady and the time dependence of the flow implicit; see Lovalenti & Brady (1993, 1995) for a detailed discussion. We also note that the boundary conditions (2.5e) assume that the introduction of the particle does not produce stresses that deform the surface of the obstacle. For a free surface in particular, this assumption only holds if the disturbance pressure is small relative to the Laplace–Young pressure jump across the interface. The deformation of the interface is much smaller than the particle size if the capillary number $Ca \equiv (\ell/a)^2 \mu u_0 / \gamma \ll 1$, γ being the surface tension of the interface S_w. This condition is satisfied in typical micron-scale applications.

The flow $\boldsymbol{u}(\boldsymbol{x})$ and the geometry of the obstacle's surface S_w have so far not been specified. To make analytical progress, we restrict our attention to particles that are small relative to the ambient flow scale, i.e. $a/\ell \ll 1$. In addition, we only consider obstacles S_w with spherical boundaries (this includes planar surfaces), and thereby draw upon the well-developed literature for two-sphere hydrodynamic interactions (Stimson & Jeffery 1926; Brenner 1961; Maude 1961). Henceforth, 'sphere' and 'particle' will both refer to the spherical particle S_p.

With no loss of generality, we take $\boldsymbol{x} = \boldsymbol{0}$ to represent the point on S_w closest to the surface of the particle S_p and $\boldsymbol{e}_\perp$ to represent the unit normal to S_w at this point, cf. figure 1. By construction, $\boldsymbol{e}_\perp$ points toward the centre of the particle, so that $\boldsymbol{x}_p = h\boldsymbol{e}_\perp$. It is also convenient to introduce cylindrical coordinates (r, θ, z) with origin at the centre of the particle as indicated in figure 1. The position of a point relative to the particle centre is then $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_p = r\boldsymbol{e}_r + z\boldsymbol{e}_z$; note that $\boldsymbol{e}_z = \boldsymbol{e}_\perp$. Although our primary focus in on the case where the particle is small relative to the obstacle ($a/R \ll 1$), the formalism that we develop here is valid for arbitrary a/R and can therefore be applied more generally to sphere–sphere interactions.

Gradients of $\boldsymbol{u}(\boldsymbol{x})$ are set by ℓ, while those of $\boldsymbol{v}(\boldsymbol{x})$ are determined by the radius of the particle a, owing to the no-slip condition (2.5c) on its surface. The background

velocity field $\boldsymbol{u}(\boldsymbol{x})$ in the vicinity of a small particle ($a/\ell \ll 1$) can be approximated by a Taylor expansion about its centre. To quadratic order in distance from the particle centre, we can write

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}|_{\boldsymbol{x}_p} + \boldsymbol{r} \cdot \nabla\boldsymbol{u}|_{\boldsymbol{x}_p} + \frac{1}{2}\boldsymbol{r}\boldsymbol{r} : \nabla\nabla\boldsymbol{u}|_{\boldsymbol{x}_p} + O\left(u_0 \frac{a^3}{\ell^3}\right). \tag{2.6}$$

Then, the surface velocity distribution $\boldsymbol{w}(\boldsymbol{x})$ in (2.5c) is

$$\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{w}^{(0)} + \boldsymbol{w}^{(1)} + \boldsymbol{w}^{(2)} + \cdots, \tag{2.7}$$

with

$$\boldsymbol{w}^{(0)} \equiv \boldsymbol{v}_p - \boldsymbol{u}|_{\boldsymbol{x}_p}, \tag{2.8a}$$

$$\boldsymbol{w}^{(1)} \equiv -\boldsymbol{r} \cdot \boldsymbol{E} + \left(\boldsymbol{\Omega}_p - \tfrac{1}{2}\boldsymbol{\omega}\right) \times \boldsymbol{r}, \tag{2.8b}$$

$$\boldsymbol{w}^{(2)} \equiv -\boldsymbol{r}\boldsymbol{r} : \boldsymbol{G}, \tag{2.8c}$$

where we define

$$\boldsymbol{E} \equiv \tfrac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}})|_{\boldsymbol{x}_p}, \quad \boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u}|_{\boldsymbol{x}_p} \quad \text{and} \quad \boldsymbol{G} \equiv \tfrac{1}{2}\nabla\nabla\boldsymbol{u}|_{\boldsymbol{x}_p}. \tag{2.9a-c}$$

Here, $\boldsymbol{E}$, $\boldsymbol{\omega}$ and $\boldsymbol{G}$ are background flow properties evaluated at the particle centre $\boldsymbol{x}_p$, *viz.* the rate-of-strain tensor ($\boldsymbol{E}$), the vorticity ($\boldsymbol{\omega}$) and a rank-3 tensor ($\boldsymbol{G}$) that quantifies the curvature of the background flow (Nadim & Stone 1991).

The hydrodynamic force on the particle under the net flow $\boldsymbol{U} = \boldsymbol{u} + \boldsymbol{v}$ is given by $\boldsymbol{F} = \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}_v \, \mathrm{d}S$, where $\boldsymbol{n}$ is the unit normal on the particle surface directed into the fluid. Due to linearity, each $\boldsymbol{w}^{(i)}$ in (2.8) can be associated with a flow ($\boldsymbol{v}^{(i)}, \boldsymbol{\sigma}_v^{(i)}$). The force on the particle is then expressible as

$$\boldsymbol{F} = \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}_v \, \mathrm{d}S = \boldsymbol{F}^{(0)} + \boldsymbol{F}^{(1)} + \boldsymbol{F}^{(2)} + \cdots, \tag{2.10}$$

where $\boldsymbol{F}^{(i)} = \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}_v^{(i)} \, \mathrm{d}S$ is the force associated with the surface velocity distribution $\boldsymbol{w}^{(i)}$ in (2.8). Thus, $\boldsymbol{F}^{(0)}$ is the hydrodynamic force due to the translation of the sphere relative to the mean local background velocity, $\boldsymbol{F}^{(1)}$ is due to the background velocity gradient and the rotation of the sphere and $\boldsymbol{F}^{(2)}$ is the contribution of the curvature of the background velocity field in the vicinity of the particle. In the absence of the boundary S_w (i.e. in an infinite medium) $\boldsymbol{F}^{(0)}$ reduces to the Stokes drag, while $\boldsymbol{F}^{(2)}$ becomes the Faxén contribution to the force, proportional to $\nabla^2\boldsymbol{u}$. By contrast, the $\boldsymbol{F}^{(1)}$ contribution, which depends on gradient of the background velocity, is identically zero for a sphere in an unbounded fluid (Happel & Brenner 1965; Batchelor 1970). However, the introduction of the boundary S_w breaks the symmetry of the disturbance flow $\boldsymbol{v}$ around the sphere, making $\boldsymbol{F}^{(1)}$ generally non-zero, as we show in subsequent sections.

If we use the representative scale v_p for the translational speed of the particle and take length scales ℓ and a, respectively, to measure gradients of $\boldsymbol{u}$ and $\boldsymbol{v}$, we obtain from (2.8) and (2.10) that the different force contributions scale as

$$\boldsymbol{F}^{(0)} = O(\mu a(v_p - u_0)), \quad \boldsymbol{F}^{(1)} = O\left(\mu a u_0 \frac{a}{\ell}\right) \quad \text{and} \quad \boldsymbol{F}^{(2)} = O\left(\mu a u_0 \left(\frac{a}{\ell}\right)^2\right). \tag{2.11a-c}$$

The force on a sphere moving in free space in the Stokes flow limit is described exactly by a combination of the Stokes drag $\boldsymbol{F}^{(0)}$ and the Faxén contribution $\boldsymbol{F}^{(2)}$, i.e. Faxén's law. In the presence of a nearby boundary, however, $\boldsymbol{F}^{(1)}$ is non-zero and can, in particular, exceed $\boldsymbol{F}^{(2)}$ since it is nominally $O(\ell/a)$ greater than the Faxén force contribution.

Here, we seek to develop analytical expressions for the force on the particle for arbitrary separations h (relative to a) between the particle and the obstacle. We further focus our attention on the force normal to the obstacle, which, as we will show, can be obtained by exploiting the geometric symmetry. The computations for the force parallel to the wall are beyond the focus of this paper, but can be obtained using the techniques presented in the following sections.

3. A reciprocal relation for the force normal to the obstacle

In this paper, we focus on the hydrodynamic force normal to the surface S_w of the obstacle, $F_z = \boldsymbol{F} \cdot \boldsymbol{e}_z$, for which we use the reciprocal theorem for Stokes flows. To this effect, we introduce as a model problem the Stokes flow due to a spherical particle translating normal to the surface of a stationary spherical obstacle (either no slip or no stress) at a velocity V' in a quiescent background fluid of viscosity μ. As in the main problem introduced in §2, we take the radius of the spherical particle to be a, the radius of the obstacle as R and the perpendicular distance of the particle's centre to the surface of the obstacle as h.

It is convenient to describe the model problem in dimensionless form using the characteristic scales a, V' and $\mu V'/a$, respectively, to scale distances, velocities and stresses. Defining $\tilde{\boldsymbol{r}} = \boldsymbol{r}/a$, and dimensionless model velocity and stress fields by $\boldsymbol{v}'$ and $\boldsymbol{\sigma}'$, respectively, the model problem satisfies

$$\nabla \cdot \boldsymbol{\sigma}' = \boldsymbol{0} \quad \text{and} \quad \nabla \cdot \boldsymbol{v}' = 0, \qquad (3.1a,b)$$

$$\boldsymbol{v}' = \boldsymbol{e}_z \quad \text{on } S_p, \qquad (3.1c)$$

$$|\boldsymbol{v}'| \to 0 \quad \text{as } |\tilde{\boldsymbol{r}}| \to \infty \quad \text{and either} \qquad (3.1d)$$

$$\left.\begin{array}{ll} \text{BC I:} & \boldsymbol{v}' = \boldsymbol{0} \quad \text{or} \\ \text{BC II:} & \boldsymbol{v}' \cdot \boldsymbol{m} = 0, \quad (\boldsymbol{m} \cdot \boldsymbol{\sigma}') \times \boldsymbol{m} = \boldsymbol{0} \end{array}\right\} \quad \text{on } S_w. \qquad (3.1e)$$

Solutions to the model problem (both for the no-slip and stress-free cases) depend on the dimensionless geometric ratios (i) h/a, which quantifies the separation distance of the particle from the surface S_w of obstacle and (ii) the curvature ratio a/R. Exact solutions to the model problem can be constructed for arbitrary h/a and a/R using known results (Brenner 1961; Maude 1961; Adamczyk, Adamczyk & Van de Ven 1983) and are summarized in appendix A. An approach similar to the one used in this paper was adopted to evaluate wall effects on the motion of a sphere in a viscoelastic fluid (Becker, McKinley & Stone 1996).

Note that while the model flow $(\boldsymbol{v}', \boldsymbol{\sigma}')$ is expressed in dimensionless form, we continue to express the main disturbance flow $(\boldsymbol{v}, \boldsymbol{\sigma}_v)$ in dimensional terms due to its dependence on the arbitrary background flow $\boldsymbol{u}(\boldsymbol{x})$. For the two Stokes flows $\boldsymbol{v}$ and $\boldsymbol{v}'$ the reciprocal theorem has the form

$$\int_{S_p+S_w+S_\infty} \boldsymbol{n} \cdot \boldsymbol{\sigma}_v \cdot \boldsymbol{v}' \, \mathrm{d}S = \frac{\mu}{a} \int_{S_p+S_w+S_\infty} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{v} \, \mathrm{d}S, \qquad (3.2)$$

where S_∞ refers to a bounding surface at infinity and $\boldsymbol{n}$ is the unit normal directed into the fluid. The factor μ/a in (3.2) is due to the definition of the dimensionless stress

tensor $\boldsymbol{\sigma}'$ in the model problem. The integral over S_∞ vanishes since $\boldsymbol{v}$ and $\boldsymbol{v}'$ decay to zero as $|\boldsymbol{r}| \to \infty$. Additionally, both $(\boldsymbol{v}, \boldsymbol{\sigma}_v)$ and $(\boldsymbol{v}', \boldsymbol{\sigma}')$ either have zero velocities on a rigid wall, or vanishing tangential stress and normal velocity components at a plane interface, due to which the surface integrals on S_w in (3.2) are identically zero. On applying the boundary conditions for $\boldsymbol{v}$ and $\boldsymbol{v}'$ on S_p, the reciprocal relationship gives the normal component of the hydrodynamic force on the sphere as

$$F_z = \boldsymbol{e}_z \cdot \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}_v \, \mathrm{d}S = \mu a \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{w}(\boldsymbol{x}) \, \mathrm{d}\tilde{S}, \tag{3.3}$$

where $\mathrm{d}\tilde{S} \equiv a^{-2} \, \mathrm{d}S$ is a dimensionless differential area element and we recall that $\boldsymbol{e}_z = \boldsymbol{e}_\perp$.

Using the decomposition of $\boldsymbol{w}(\boldsymbol{x})$ in (2.7) we can write

$$F_z^{(0)} = \mu a \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{w}^{(0)} \, \mathrm{d}\tilde{S} = \mu a \boldsymbol{w}^{(0)} \cdot \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \, \mathrm{d}\tilde{S}, \tag{3.4a}$$

$$F_z^{(1)} = \mu a \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{w}^{(1)} \, \mathrm{d}\tilde{S}$$

$$= \mu a^2 \left\{ -\boldsymbol{E} : \int_{S_p} \tilde{\boldsymbol{r}} (\boldsymbol{n} \cdot \boldsymbol{\sigma}') \, \mathrm{d}\tilde{S} + \left(\boldsymbol{\Omega}_p - \frac{1}{2}\boldsymbol{\omega} \right) \cdot \int_{S_p} \tilde{\boldsymbol{r}} \times (\boldsymbol{n} \cdot \boldsymbol{\sigma}') \, \mathrm{d}\tilde{S} \right\} \quad \text{and} \tag{3.4b}$$

$$F_z^{(2)} = \mu a \int_{S_p} \boldsymbol{n} \cdot \boldsymbol{\sigma}' \cdot \boldsymbol{w}^{(2)} \, \mathrm{d}\tilde{S} = -\mu a^3 \boldsymbol{G} \vdots \int_{S_p} \tilde{\boldsymbol{r}} \tilde{\boldsymbol{r}} (\boldsymbol{n} \cdot \boldsymbol{\sigma}') \, \mathrm{d}\tilde{S}. \tag{3.4c}$$

The force contribution $F_z^{(i)}$ is therefore linearly related to the *i*th moment of the surface traction $\boldsymbol{n} \cdot \boldsymbol{\sigma}'$ on the sphere in the model problem.

The above expressions involve only integrals on the particle surface S_p. Thus, they are generally valid for an arbitrary particle suspended in a quadratic flow as long as the model flow $(\boldsymbol{v}', \boldsymbol{\sigma}')$ satisfies (3.1), although we note that the solution to the corresponding model problem may not be known for arbitrary geometries. We also remark that a more general application of the reciprocal theorem, with the appropriate choice of model problem, can be used to compute the full mobility matrix for a sphere near a boundary; see e.g. Becker *et al.* (1996), Haber & Brenner (1999). While solutions to the relevant model problems (wall-parallel translation and rotation of a sphere) are known (Dean & O'Neill 1963; O'Neill 1964), we do not attempt these more involved calculations here. Rather, we focus on the normal force component, which can be computed explicitly by utilizing the symmetry in the model problem (3.1).

3.1. *Axisymmetry of the model problem*

To exploit the spherical symmetry, we parametrize the surface of the particle in the *rz* section through using intrinsic coordinates, see e.g. Stimson & Jeffery (1926), Happel & Brenner (1965). As shown in figure 2, we define *s* as the dimensionless arc length in the *rz* plane (non-dimensionalized using the particle radius *a*), such that $s = 0$ and $s = \pi$ correspond, respectively, to the points on the sphere farthest and closest to S_w, i.e.

$$s = 0 \quad \text{at} \ (\tilde{r} = 0, \ \tilde{z} = +1) \quad \text{and} \tag{3.5a}$$

$$s = \pi \quad \text{at} \ (\tilde{r} = 0, \ \tilde{z} = -1). \tag{3.5b}$$
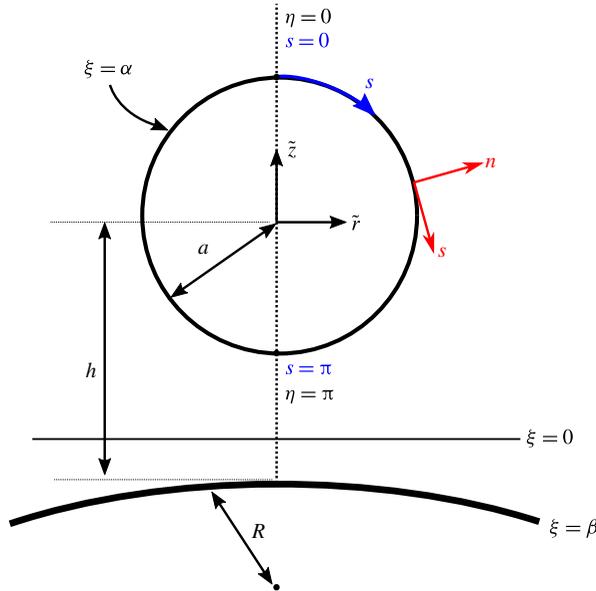
FIGURE 2. (Colour online) Parametrization of the particle surface in polar coordinates $(\tilde{r}, \tilde{z})$, showing intrinsic coordinates (s, n). The evaluation of the force ultimately involves bipolar coordinates (ξ, η), which are introduced in §4. Here, α and β are dimensionless quantities relevant to the bipolar coordinate system that are related to h/a and R/a and are defined in §4.

The surface of the particle is then described by rotating the curve defined by $0 < s < \pi$ about the z-axis through an azimuthal angle θ of 2π. Under the change of coordinates, surface integrals transform as

$$\int_{S_p} f \, d\tilde{S} = \int_0^\pi \int_0^{2\pi} f \, \tilde{r} \, d\theta \, ds \qquad (3.6)$$

for an arbitrary scalar field f defined on the spherical particle surface S_p.

Due to the boundary conditions and the geometry (spherical particle and obstacle surfaces), the model problem is axisymmetric about the z-axis. The θ integration in the surface integrals of (3.4), when written using (3.6), can then be immediately evaluated without explicit knowledge of σ'. Defining

$$\overline{w}^{(i)} \equiv \frac{1}{2\pi} \int_0^{2\pi} \left\{ (e_r \cdot w^{(i)}) e_r + (e_z \cdot w^{(i)}) e_z \right\} d\theta, \quad \text{with } i \in \{0, 1, 2\}, \qquad (3.7)$$

one can write

$$F_z^{(i)} = 2\pi\mu a \int_0^\pi n \cdot \sigma' \cdot \overline{w}^{(i)} \, \tilde{r} \, ds \quad \text{on } S_p. \qquad (3.8)$$

Substituting (2.8) into (3.7) and using the symmetries of the tensors $\boldsymbol{E}$ and $\boldsymbol{G}$, we obtain

$$\overline{w}^{(0)} = w_z^{(0)} e_z, \qquad (3.9a)$$

$$\overline{w}^{(1)} = -aE_{zz} \left(-\frac{\tilde{r}}{2} e_r + \tilde{z} e_z \right) \quad \text{and} \qquad (3.9b)$$

$$\overline{\boldsymbol{w}}^{(2)} = -a^2 G_{zzz} \left\{ -\tilde{r}\tilde{z}\boldsymbol{e}_r + \left(\tilde{z}^2 - \frac{\tilde{r}^2}{2} \right) \boldsymbol{e}_z \right\} - a^2 G_{jjz} \frac{\tilde{r}^2}{2} \boldsymbol{e}_z \quad \text{with } j \in \{1, 2, 3\}, \quad (3.9c)$$

where summation over j is implied in G_{jjz}. We observe that the normal force only depends on certain components of the tensors $\boldsymbol{E}$ and $\boldsymbol{G}$, in particular $E_{zz} = \boldsymbol{e}_z \boldsymbol{e}_z : \boldsymbol{\nabla}\boldsymbol{u}|_{x_p}$, $G_{zzz} = (1/2)\boldsymbol{e}_z\boldsymbol{e}_z\boldsymbol{e}_z \vdots \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}|_{x_p}$ and $G_{jjz} = (1/2)\boldsymbol{e}_z \cdot \boldsymbol{\nabla}^2\boldsymbol{u}|_{x_p}$. We note also that $F_z^{(1)}$ is independent of the angular velocity $\boldsymbol{\Omega}_p$ of the sphere and the vorticity $\boldsymbol{\omega}$ of the background flow as a consequence of the rotational symmetry of the particle about the z axis.

The unit tangent and outward normal vectors $\boldsymbol{s}$ and $\boldsymbol{n}$, respectively, to the surface of the sphere are by definition

$$\boldsymbol{s} \equiv \frac{\mathrm{d}\tilde{\boldsymbol{r}}}{\mathrm{d}s} = \frac{\mathrm{d}\tilde{r}}{\mathrm{d}s}\boldsymbol{e}_r + \frac{\mathrm{d}\tilde{z}}{\mathrm{d}s}\boldsymbol{e}_z \quad \text{and} \quad \boldsymbol{n} \equiv \boldsymbol{s} \times \boldsymbol{e}_\theta = -\frac{\mathrm{d}\tilde{z}}{\mathrm{d}s}\boldsymbol{e}_r + \frac{\mathrm{d}\tilde{r}}{\mathrm{d}s}\boldsymbol{e}_z. \quad (3.10a,b)$$

Under the axisymmetry of the model flow, we can then write

$$\boldsymbol{n} \cdot \boldsymbol{\sigma}' = -p'\boldsymbol{n} + \omega'\boldsymbol{s} + \left(\frac{v_r'}{\tilde{r}}\boldsymbol{n} - 2\frac{\partial\boldsymbol{v}'}{\partial s} \times \boldsymbol{e}_\theta \right) \quad \text{on } S_p, \quad (3.11)$$

where p' is the dimensionless pressure (in units of $\mu V'/a$) and $\omega' = \boldsymbol{e}_\theta \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}')$ is the dimensionless vorticity field (in units of V'/a) of the model problem (Happel & Brenner 1965). The no-slip condition on the sphere in the model problem ($\boldsymbol{v}' = \boldsymbol{e}_z$ on S_p, cf. equation 3.1c), ensures that the term in parentheses in (3.11) is identically zero. Substituting (3.10) and (3.11) into (3.8) then results in

$$F_z^{(0)} = 2\pi a w_z^{(0)} \int_0^\pi \left\{ -p'\frac{\partial}{\partial s}\left(\frac{\tilde{r}^2}{2} \right) + \omega'\tilde{r}\frac{\partial\tilde{z}}{\partial s} \right\} \mathrm{d}s, \quad (3.12a)$$

$$F_z^{(1)} = -2\pi a^2 E_{zz} \int_0^\pi \left\{ -p'\frac{\partial}{\partial s}\left(\frac{\tilde{r}^2\tilde{z}}{2} \right) + \omega'\tilde{r}\frac{\partial}{\partial s}\left(\frac{\tilde{z}^2}{2} - \frac{\tilde{r}^2}{4} \right) \right\} \mathrm{d}s, \quad (3.12b)$$

$$F_z^{(2)} = -2\pi a^3 G_{zzz} \int_0^\pi \left\{ -p'\frac{\partial}{\partial s}\left(\frac{\tilde{r}^2\tilde{z}^2}{2} - \frac{\tilde{r}^4}{8} \right) + \omega'\left(\frac{\tilde{r}\tilde{z}}{2}\frac{\partial}{\partial s}\left(\frac{\tilde{z}^2 - \tilde{r}^2}{2} \right) - \frac{\tilde{r}^3}{2}\frac{\partial\tilde{z}}{\partial s} \right) \right\} \mathrm{d}s$$
$$- 2\pi a^3 G_{jjz} \int_0^\pi \left\{ -p'\frac{\partial}{\partial s}\left(\frac{\tilde{r}^4}{8} \right) + \omega'\frac{\tilde{r}^3}{2}\frac{\partial\tilde{z}}{\partial s} \right\} \mathrm{d}s, \quad (3.12c)$$

where it is understood that the kernels of the integrals are evaluated on S_p.

Through an integration by parts, integrals involving p' can be recast in terms of integrals involving $\partial p'/\partial s$; boundary terms are identically zero since $\tilde{r} = 0$ for both $s = 0$ and $s = \pi$. Writing $\partial p'/\partial s = \boldsymbol{s} \cdot \boldsymbol{\nabla}p'$ and noting that the momentum equation for the axisymmetric dimensionless model problem can be expressed as $\boldsymbol{\nabla}p' = -\boldsymbol{\nabla} \times (\omega'\boldsymbol{e}_\theta)$, we write

$$\frac{\partial p'}{\partial s} = \frac{1}{\tilde{r}}\frac{\partial(\tilde{r}\omega')}{\partial n}, \quad (3.13)$$

where $\partial/\partial n = \boldsymbol{n} \cdot \boldsymbol{\nabla}$ is the derivative normal to the surface of the sphere. Thus, on eliminating the pressure p' from (3.12) in favour of the vorticity ω' using (3.13), the expressions for $F_z^{(0)}$, $F_z^{(1)}$ and $F_z^{(2)}$ become

$$F_z^{(0)} = \pi\mu a w_z^{(0)} \int_0^\pi \tilde{r}^3\frac{\partial}{\partial n}\left(\frac{\omega'}{\tilde{r}} \right) \mathrm{d}s, \quad (3.14a)$$

$$F_z^{(1)} = -\pi\mu a^2 E_{zz} \int_0^\pi \tilde{r}^3 \tilde{z}^2 \frac{\partial}{\partial n} \left(\frac{1}{\tilde{z}} \frac{\omega'}{\tilde{r}} \right) \mathrm{d}s \quad \text{and} \tag{3.14b}$$

$$F_z^{(2)} = -\pi\mu a^3 \left[G_{zzz} \int_0^\pi \left\{ \tilde{r}^3 \tilde{z}^4 \frac{\partial}{\partial n} \left(\frac{1}{\tilde{z}^2} \frac{\omega'}{r} \right) - \frac{r^7}{4} \frac{\partial}{\partial n} \left(\frac{\omega'}{\tilde{r}^3} \right) \right\} \mathrm{d}s \right.$$

$$\left. + G_{jjz} \int_0^\pi \frac{\tilde{r}^7}{4} \frac{\partial}{\partial n} \left(\frac{\omega'}{\tilde{r}^3} \right) \mathrm{d}s \right]. \tag{3.14c}$$

The force normal to the surface of the obstacle can therefore be computed using only the vorticity field of the model problem. The dependence on μ, a and the background flow has been scaled out explicitly in (3.14), so that the integrals themselves are dimensionless and only depend on the dimensionless ratios a/h and a/R.

Note that since $F_z^{(0)}$ depends on the translational velocity of the sphere relative to the fluid, it is related to the force on the particle in the model problem F_z' as $F_z^{(0)} = F_z'(w_z^{(0)}/V')$. This contribution to the force is already well known (Brenner 1961; Cox & Brenner 1967) for a translating sphere in a quiescent fluid, but is obtained here as the leading term of a more general expansion of the background flow. The expressions for $F_z^{(1)}$ and $F_z^{(2)}$ have a similar structure to $F_z^{(0)}$ and are calculated in the next section.

4. Evaluation of the normal force on the sphere

4.1. *Vorticity field in the model problem*

The normal force on the sphere, equation (3.14), can be computed given the vorticity field in the model problem, which corresponds to the translation of a spherical particle normal to a stationary spherical obstacle (on which the flow satisfies either no-slip or stress-free conditions) in a quiescent fluid. For arbitrary a/h and a/R, the solution to the model problem can be expressed using bipolar coordinates (ξ, η), in which the particle surface S_p and the obstacle surface S_w are delineated by the coordinate lines $\xi = \alpha$ and $\xi = \beta$, respectively (see figure 2), where (Adamczyk *et al.* 1983)

$$\alpha \equiv \cosh^{-1} \left(\frac{\left(\frac{h}{a} + \frac{R}{a} \right)^2 - \left(\frac{R}{a} \right)^2 + 1}{2 \left(\frac{h}{a} + \frac{R}{a} \right)} \right) \quad \text{and} \quad \frac{\sinh\beta}{\sinh\alpha} \equiv -\frac{a}{R}. \tag{4.1a,b}$$

Taken together, the values of α and β specify both a/h and a/R and therefore uniquely determine the geometry of the problem. The coordinates (ξ, η) are related to the cylindrical coordinates (r, z) as (Stimson & Jeffery 1926)

$$\tilde{r} = \frac{\sinh\alpha \sin\eta}{\cosh\xi - \cos\eta} \quad \text{and} \quad \tilde{z} = \frac{\sinh\alpha \sinh\xi}{\cosh\xi - \cos\eta} - \cosh\alpha. \tag{4.2a,b}$$

Thus, ξ takes values in the range $[\beta, \alpha]$, where we take $\beta < 0$ and $\alpha > 0$ without loss of generality, cf. (4.1). The coordinate η lies in $[0, \pi]$; on the particle S_p, $\eta = 0 \Longleftrightarrow s = 0$ and $\eta = \pi \Longleftrightarrow s = \pi$, cf. figure 2. We note that for a fixed curvature ratio a/R, the limit $\alpha \to 0$ corresponds to the case where the surfaces are touching ($h \to a$) and $\alpha \to \infty$ to large separations ($h/a \to \infty$).

The model vorticity field ω' can be described in these coordinates, for arbitrary α and β, by first defining the functions

$$
\begin{aligned}
U_n(\xi) &= a_n \cosh(n - \tfrac{1}{2})\xi + b_n \sinh(n - \tfrac{1}{2})\xi \\
&\quad + c_n \cosh(n + \tfrac{3}{2})\xi + d_n \sinh(n + \tfrac{3}{2})\xi \quad \text{and}
\end{aligned}
\tag{4.3a}
$$

$$
V_n(\tau) = \frac{P_{n-1}(\tau) - P_{n+1}(\tau)}{2n + 1},
\tag{4.3b}
$$

where $\tau \equiv \cos \eta$ and P_n denotes a Legendre polynomial of order n. The dimensionless coefficients a_n, b_n, c_n and d_n are determined by the boundary conditions on S_p and S_w and depend on α and β; explicit formulas are given in appendix A. The dimensionless quantity $\omega'/\tilde{r}$ appearing in the expressions for F_z can then be written as (Stimson & Jeffery 1926; Brenner 1961)

$$
\frac{\omega'}{\tilde{r}} = \frac{g(\xi, \tau)}{(1 - \tau^2)\sinh^4 \alpha},
\tag{4.4}
$$

where

$$
\begin{aligned}
g(\xi, \tau) &= (\cosh \xi - \tau)^{5/2} \sum_{n=0}^{\infty} \Bigg[V_n(\tau) \Bigg\{ \frac{\mathrm{d}^2 U_n(\xi)}{\mathrm{d}\xi^2} - \frac{2 \sinh \xi}{\cosh \xi - \tau} \frac{\mathrm{d}U_n(\xi)}{\mathrm{d}\xi} \\
&\quad + \frac{3}{4} \frac{\cosh \xi + 3\tau}{\cosh \xi - \tau} U_n(\xi) \Bigg\} \\
&\quad + (1 - \tau^2) U_n(\xi) \Bigg\{ \frac{\mathrm{d}^2 V_n(\tau)}{\mathrm{d}\tau^2} + \frac{2}{\cosh \xi - \tau} \frac{\mathrm{d}V_n(\tau)}{\mathrm{d}\tau} \Bigg\} \Bigg].
\end{aligned}
\tag{4.5}
$$

This completes the description of the model vorticity field ω' required to calculate the normal force on the particle using (3.14). We remark that the curvature ratio a/R and boundary conditions on both surfaces enter the problem through the coefficients a_n, b_n, c_n and d_n, which depend on both a/h and a/R (via α and β). In the limit $a/R \to 0$ (planar S_w), we find from (4.1) that $\alpha \to \cosh^{-1}(h/a)$ and $\beta \to 0$. The solutions to the model problems (both no-slip and stress-free cases) in this limit are given by Brenner (1961) for arbitrary h/a (i.e. arbitrary α). For the general case of a curved obstacle ($\beta \neq 0$) and no-slip surfaces (BC I), the appropriate expressions for the coefficients a_n, b_n, c_n and d_n can be constructed using two-sphere solutions (Maude 1961; Adamczyk et al. 1983) and depend on both α and β (see appendix A). The coefficients corresponding to a stress-free curved obstacle S_w ($\beta \neq 0$; BC II) have been computed in appendix A.

4.2. Expressions for the force

To utilize the expression (4.4) for ω' in the integrals of (3.14), it is necessary to transform $\mathrm{d}s$ and $\mathrm{d}n$ to bipolar coordinates. Writing $\mathrm{d}n = \boldsymbol{n} \cdot \mathrm{d}\boldsymbol{r}$ and $\mathrm{d}s = \boldsymbol{s} \cdot \mathrm{d}\boldsymbol{r}$ in (3.10) and using the transformation (4.2), we obtain

$$
(\mathrm{d}n, \ \mathrm{d}s) = \frac{\sinh \alpha}{\cosh \alpha - \cos \eta} (-\mathrm{d}\xi, \ \mathrm{d}\eta) \quad \text{on } \xi = \alpha,
\tag{4.6}
$$

so that

$$
\int_0^\pi \frac{\partial f}{\partial n} \, \mathrm{d}s = -\int_0^\pi \frac{\partial f}{\partial \xi} \, \mathrm{d}\eta = -\int_{-1}^1 \frac{\partial f}{\partial \xi} \frac{\mathrm{d}\tau}{\sqrt{1 - \tau^2}} \quad \text{on } S_p,
\tag{4.7}
$$

for any function f. Substituting the expressions (4.2)–(4.7) for the coordinate transformation $(r, z) \mapsto (\xi, \eta)$ and the model vorticity field ω' into (3.14), we obtain

$$F_z^{(0)} = -6\pi\mu a w_z^{(0)} \mathcal{I}_0, \tag{4.8a}$$

$$F_z^{(1)} = -6\pi\mu a^2 E_{zz} \left(\mathcal{I}_0 \cosh \alpha - \mathcal{I}_1 \right) \quad \text{and} \tag{4.8b}$$

$$F_z^{(2)} = 6\pi\mu a^3 \left\{ G_{zzz} \left(\mathcal{I}_0 \cosh^2 \alpha - 2\mathcal{I}_1 \cosh \alpha + \mathcal{I}_2 - \mathcal{J} \right) + G_{jjz} \mathcal{J} \right\}, \tag{4.8c}$$

where

$$\mathcal{I}_k \equiv \frac{1}{6} \int_{-1}^{1} \frac{(\sinh \xi)^{3k-1}}{(\cosh \xi - \tau)^{2k+3}} \frac{\partial}{\partial \xi} \left\{ \left(\frac{\cosh \xi - \tau}{\sinh \xi} \right)^k g(\xi, \tau) \right\} d\tau \bigg|_{\xi=\alpha} \quad \text{and} \tag{4.9a}$$

$$\mathcal{J} \equiv \frac{1}{24} \int_{-1}^{1} \frac{(1 - \tau^2) \sinh \xi}{(\cosh \xi - \tau)^7} \frac{\partial}{\partial \xi} \left\{ (\cosh \xi - \tau)^2 g(\xi, \tau) \right\} d\tau \bigg|_{\xi=\alpha}. \tag{4.9b}$$

The integrals $\mathcal{I}_k$ and $\mathcal{J}$ are functions of α and β and involve the coefficients a_n, b_n, c_n and d_n, which enter through $g(\xi, \tau)$ in (4.5).

The explicit evaluation of $\mathcal{I}_k$ and $\mathcal{J}$ utilizes some properties of the Legendre polynomials that appear in $g(\xi, \tau)$; the details are given in appendix B. The integration ultimately results in expressions for the force in terms of the coefficients a_n, b_n, c_n and d_n of the model flow, which we express as

$$F_z^{(0)} = -6\pi\mu a w_z^{(0)} \mathcal{A}, \tag{4.10a}$$

$$F_z^{(1)} = -6\pi\mu a^2 E_{zz} \mathcal{B}, \tag{4.10b}$$

$$F_z^{(3)} = 6\pi\mu a^3 \left(G_{zzz} \mathcal{C} + G_{jjz} \mathcal{D} \right), \tag{4.10c}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are dimensionless hydrodynamic resistances given by

$$\mathcal{A} = -\frac{\sqrt{2}}{3 \sinh \alpha} \sum_{n=0}^{\infty} \{a_n + b_n + c_n + d_n\}, \tag{4.11a}$$

$$\mathcal{B} = \mathcal{A} \cosh \alpha + \frac{\sqrt{2}}{3} \sum_{n=0}^{\infty} \{(2n-1)(a_n + b_n) + (2n+3)(c_n + d_n)\}, \tag{4.11b}$$

$$\mathcal{C} = 2\mathcal{B} \cosh \alpha - \mathcal{A} \cosh^2 \alpha - \frac{\sqrt{2} \sinh \alpha}{9} \sum_{n=0}^{\infty} \left[n(n+1)(a_n + b_n + c_n + d_n) \right.$$
$$\left. + \{(2n-1)^2(a_n + b_n) + (2n+3)^2(c_n + d_n)\} \right], \tag{4.11c}$$

$$\mathcal{D} = \frac{\mathcal{A}}{3} - \frac{\mathcal{C}}{5} + \frac{2\sqrt{2}}{45} \sum_{n=0}^{\infty} \left[(2(2n+1) - \coth \alpha)\{(a_n - b_n)e^{\alpha} + (c_n - d_n)e^{-\alpha}\} \right.$$
$$\left. - 5\{(a_n - b_n)e^{\alpha} - (c_n - d_n)e^{-\alpha}\} \right] e^{-(2n+1)\alpha}. \tag{4.11d}$$

The dimensionless resistances $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are positive and can be written as functions of α and β by substituting the analytical expressions for a_n, b_n, c_n and d_n (see appendix A).

Thus, we have obtained general expressions for the hydrodynamic resistances of a spherical particle near a stationary spherical obstacle suspended in a locally quadratic

background flow. The component of the hydrodynamic force on the sphere normal to the obstacle, accurate up to the quadratic moment of the flow, is therefore

$$
F_\perp \equiv \boldsymbol{F} \cdot \boldsymbol{e}_\perp \;=\; 6\pi\mu a \Bigg[\Big\{ -\mathcal{A}\,(\boldsymbol{v}_p - \boldsymbol{u}) - a\,\mathcal{B}\,(\boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}\boldsymbol{u})
$$
$$
+\frac{a^2}{2}\mathcal{C}\,(\boldsymbol{e}_\perp\boldsymbol{e}_\perp : \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}) + \frac{a^2}{2}\mathcal{D}\,\nabla^2\boldsymbol{u} \Big\} \cdot \boldsymbol{e}_\perp \Bigg]_{x_p}. \tag{4.12}
$$

5. Discussion

The general expression for the normal force (4.12) shows that there are four distinct contributions up to the second moment of the background flow. The first of these, proportional to $\mathcal{A}$, is due to the translation of the particle relative to the mean background flow in its vicinity, and is identical to the general expression obtained by Brenner (1961). The term proportional to $\mathcal{B}$ arises due to extensional gradients of the background flow. This contribution is identically zero for a sphere in an infinite medium (Happel & Brenner 1965; Batchelor 1970), but as we will show shortly, is generally non-zero for finite h/a and is ultimately important in determining the motion of the particle close to the obstacle. Second moments of the background flow result in two separate contributions to the force: one that is proportional to $\mathcal{C}$ and depends on the curvature of the background velocity normal to the plane $(\boldsymbol{e}_\perp\boldsymbol{e}_\perp : \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u})$ and another that is proportional to $\mathcal{D}$ and depends on $\nabla^2\boldsymbol{u}$. In the discussion that follows, the subscripts I or II will be used, when necessary, to distinguish between the cases in which the surface of the obstacle is a no-slip boundary or a no-stress interface, respectively.

We now discuss the asymptotic behaviour of the hydrodynamic resistance coefficients in the limits of large and small separations of the sphere from the obstacle. It is convenient to describe the limiting behaviour in terms of a dimensionless parameter

$$
\Delta \equiv \frac{h-a}{a} = \cosh\alpha - 1 - \frac{R}{a} + \sqrt{\sinh^2\alpha + \left(\frac{R}{a}\right)^2}, \tag{5.1}
$$

which is the minimum surface-to-surface distance relative to the radius of the particle. The hydrodynamic resistances depend on the two dimensionless parameters h/a and R/a.

Although the formalism is general for arbitrary a/R, we restrict our attention to situations with small particles $(a/R \ll 1)$. Hydrodynamic interactions are weak for very large separations with $h \gg R \gg a$ and are well described by interactions between point objects using the method of reflections (Haber & Brenner 1999). We focus here on the stronger interactions associated with separation distances satisfying $R \gg h \gtrsim a$ (including $R \gg h \gg a$), which corresponds to $|\beta| \ll \alpha$, cf. (4.1). Thus, a Taylor expansion of (4.11) about $\beta = 0$ (i.e. the limit of a planar obstacle, $a/R \to 0$) using the results of appendix A is appropriate. The leading-order forms of the resistances $\mathcal{A}_\mathrm{I}$, $\mathcal{B}_\mathrm{I}$, $\mathcal{C}_\mathrm{I}$ and $\mathcal{D}_\mathrm{I}$ (no-slip wall) and $\mathcal{A}_\mathrm{II}$, $\mathcal{B}_\mathrm{II}$, $\mathcal{C}_\mathrm{II}$ and $\mathcal{D}_\mathrm{II}$ (no-stress interface) for $|\beta| \ll 1$ are plotted in figure 3(a,b), respectively, as functions of Δ.

5.1. *Asymptotic behaviour for small and large separations*

The asymptotic behaviour of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ for large separations $\Delta \gg 1$ (in the limit $a/R \to 0$) is obtained by a term-by-term Taylor series expansion of the expressions in
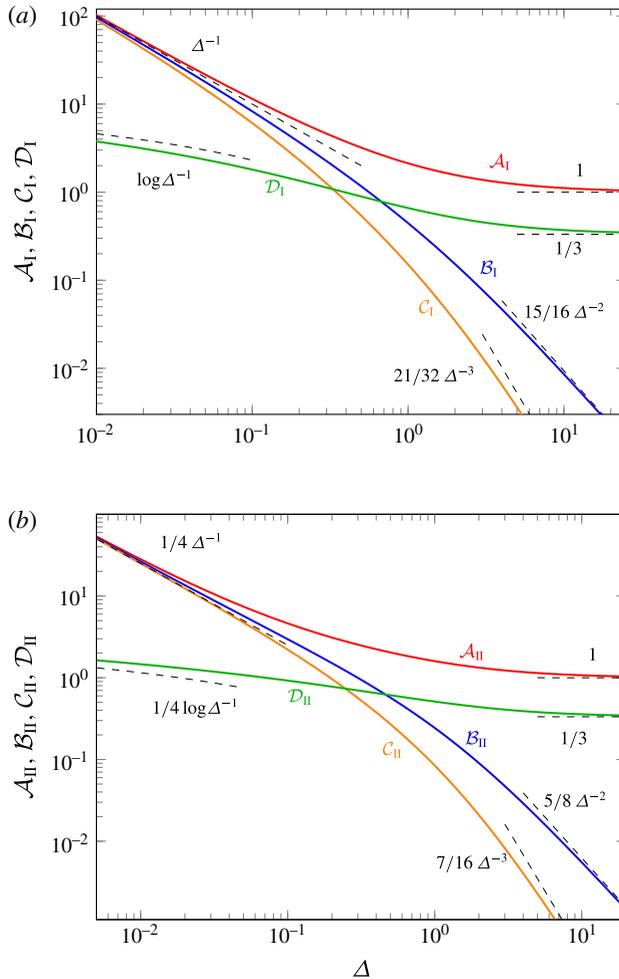
FIGURE 3. (Colour online) Resistance coefficients for the cases of an obstacle with (*a*) a solid surface and (*b*) a stress-free interface in the limit of a weakly curved obstacle ($a/R \rightarrow 0 \Longleftrightarrow \beta \rightarrow 0$). Dashed lines indicate the leading asymptotic behaviour for large and small Δ.

(C 3) for large α (note that $\Delta \gg 1 \Longrightarrow \alpha \gg 1$), and then evaluating the sum. Only the first few terms of the series are required to obtain asymptotically accurate results.

For arbitrarily large separations $\Delta \rightarrow \infty$, only $\mathcal{A}$ and $\mathcal{D}$ are non-zero, with $\mathcal{B}$ and $\mathcal{C}$ decaying algebraically to zero. Independent of the boundary condition at the wall S_w, we obtain

$$\mathcal{A} \rightarrow 1, \quad \mathcal{B} \rightarrow 0, \quad \mathcal{C} \rightarrow 0, \quad \mathcal{D} \rightarrow \tfrac{1}{3}, \quad \text{as} \ \Delta \rightarrow \infty. \quad (5.2a-d)$$

In this limit, from (4.12) we recover Faxén's law for the force on a sphere in an unbounded fluid:

$$\boldsymbol{F} = 6\pi\mu a \left[(\boldsymbol{u}|_{\boldsymbol{x}_p} - \boldsymbol{v}_p) + \frac{a^2}{6} \nabla^2 \boldsymbol{u}|_{\boldsymbol{x}_p} \right], \quad \text{as} \ \Delta \rightarrow \infty. \quad (5.3)$$

We find additionally that $\mathcal{B} \propto \Delta^{-2}$ and $\mathcal{C} \propto \Delta^{-3}$ for large $\Delta \gg 1$, with $O(1)$ prefactors that depend on the boundary conditions at S_w; specifically, we obtain

$$\mathcal{B}_{\mathrm{I}} \sim \tfrac{15}{16}\Delta^{-2}, \quad \mathcal{B}_{\mathrm{II}} \sim \tfrac{5}{8}\Delta^{-2}, \quad \mathcal{C}_{\mathrm{I}} \sim \tfrac{21}{32}\Delta^{-3} \quad \text{and} \quad \mathcal{C}_{\mathrm{II}} \sim \tfrac{7}{16}\Delta^{-3}, \quad \text{for } \Delta \gg 1. \quad (5.4a\text{--}d)$$

The above far-field expressions are in agreement with the resistances calculated using the method of reflections (Haber & Brenner 1999; Swan & Brady 2007).

For small separations ($\Delta \ll 1$), the asymptotic forms of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ cannot in general be evaluated to arbitrary order in Δ (in particular beyond $O(\Delta^{-1})$ terms) using a term-by-term expansion of (C 3). Instead, a more sophisticated approach involving matched asymptotic expansions of the infinite series in (C 3) is required (Cox & Brenner 1967), using which we obtain at leading order:

$$\mathcal{A} \sim \mathcal{B} \sim \mathcal{C} \sim \frac{K}{\Delta} \quad \text{and} \quad \mathcal{D} \sim K \log \Delta^{-1}, \quad \text{for } \Delta \ll 1. \quad (5.5a,b)$$

Here $K = K_{\mathrm{I}} = 1$ for a no-slip wall, and $K = K_{\mathrm{II}} = 1/4$ for a stress-free surface. These leading-order asymptotes in the small Δ limit can also be obtained by using the well-known description of the model flow from lubrication theory, rather than the exact solution in terms of bipolar coordinates. We verify that using the lubrication theory prediction for the model vorticity ω' (O'Neill & Stewartson 1967; Jeffrey & Onishi 1981) in the general formulas (3.14) recovers the leading-order asymptotes (5.5) obtained from analysing the full expressions in (C 3). The asymptotic forms of the resistance coefficients for large and small Δ in the limit $a/R \to 0$ are indicated in figure 3 as dashed lines. These asymptotes, along with leading-order corrections in a/R, are tabulated in table 1. The logarithmic and $O(\Delta^0)$ terms in the $\Delta \ll 1$ limit (table 1) are found using the method detailed in Cox & Brenner (1967), with the $O(\Delta^0)$ terms obtained from a numerical integration of known analytic functions.

While the analysis formally applies to spherical particles near spherical obstacles, some general quantitative conclusions can be drawn for non-spherical obstacles. Consider obstacles that are only locally spherical, i.e. the principle radii of curvature are both locally equal to R (cf. figure 1). If (i) the particle radius and its distance from the obstacle are small ($a \lesssim h \ll R$) and (ii) gradients in the curvature of the obstacle's surface occur over length scales of $O(R)$, these curvature gradients have a small effect on the normal force since the disturbance flow due to the particle decays at most over a characteristic length scale of $h \ll R$. In this case, the results in table 1 remain applicable to first approximation. For $h \gtrsim O(R)$, the disturbance flow caused by the particle must conform to the detailed shape of the obstacle boundary and a local description does not suffice. Similar arguments apply if the principle radii of curvature of the obstacle, say R_1 and R_2, are distinct, although it is noted that for a small particle that is close to the obstacle surface ($a \lesssim h \ll R_1$, $a \lesssim h \ll R_2$, $R_1 \neq R_2$), only the terms independent of a/R in table 1 are meaningful.

5.2. *Force on a stationary particle near an obstacle*

We now compute the external force that must be applied in order to hold a particle stationary on the symmetry axis of a normally incident stagnation flow near an obstacle (figure 4). By symmetry, the only component of the external force required to keep the particle stationary is in the direction normal to the surface. In the near-field limit ($\Delta \ll 1$) the background flow may be alternately expressed using a Taylor expansion about the point on the obstacle closest to the particle surface, $\boldsymbol{x} = \boldsymbol{0}$,

Dimensionless resistance coefficients	$\Delta \ll 1$	$\dfrac{R}{a} \gg \Delta \gg 1$
$\mathcal{A}_{\mathrm{I}}$	$\Delta^{-1} + \dfrac{1}{5}\log\Delta^{-1} + 0.9713 + \dfrac{a}{R}\left(-2\Delta^{-1} + \dfrac{4}{5}\log\Delta^{-1} - 0.0070\right)$	$1 + \dfrac{9}{8}\Delta^{-1} - \dfrac{9}{16}\dfrac{a}{R}$
$\mathcal{B}_{\mathrm{I}}$	$\Delta^{-1} - \dfrac{4}{5}\log\Delta^{-1} + 0.3070 + \dfrac{a}{R}\left(-2\Delta^{-1} + \dfrac{24}{5}\log\Delta^{-1} - 5.0378\right)$	$\dfrac{15}{16}\Delta^{-2}$
$\mathcal{C}_{\mathrm{I}}$	$\Delta^{-1} - \dfrac{14}{5}\log\Delta^{-1} + 3.7929 + \dfrac{a}{R}\left(-2\Delta^{-1} + \dfrac{59}{5}\log\Delta^{-1} - 21.9088\right)$	$\dfrac{21}{32}\Delta^{-3} + \dfrac{21}{64}\dfrac{a}{R}\Delta^{-2}$
$\mathcal{D}_{\mathrm{I}}$	$\log\Delta^{-1} - 0.9208 + \dfrac{a}{R}\left(-3\log\Delta^{-1} + 5.7157\right)$	$\dfrac{1}{3} + \dfrac{3}{8}\Delta^{-1} - \dfrac{3}{16}\dfrac{a}{R}$
$\mathcal{A}_{\mathrm{II}}$	$\dfrac{1}{4}\Delta^{-1} + \dfrac{9}{20}\log\Delta^{-1} + 1.0337 + \dfrac{a}{R}\left(-\dfrac{1}{2}\Delta^{-1} - \dfrac{9}{20}\log\Delta^{-1} + 0.4300\right)$	$1 + \dfrac{3}{4}\Delta^{-1} - \dfrac{3}{8}\dfrac{a}{R}$
$\mathcal{B}_{\mathrm{II}}$	$\dfrac{1}{4}\Delta^{-1} + \dfrac{9}{20}\log\Delta^{-1} - 0.7550 + \dfrac{a}{R}\left(-\dfrac{1}{2}\Delta^{-1} + \dfrac{1}{20}\log\Delta^{-1} + 1.2243\right)$	$\dfrac{5}{8}\Delta^{-2}$
$\mathcal{C}_{\mathrm{II}}$	$\dfrac{1}{4}\Delta^{-1} + \dfrac{1}{5}\log\Delta^{-1} - 1.0460 + \dfrac{a}{R}\left(-\dfrac{1}{2}\Delta^{-1} + \dfrac{13}{10}\log\Delta^{-1} - 0.4161\right)$	$\dfrac{7}{16}\Delta^{-3} + \dfrac{7}{32}\dfrac{a}{R}\Delta^{-2}$
$\mathcal{D}_{\mathrm{II}}$	$\dfrac{1}{4}\log\Delta^{-1} + 0.3066 + \dfrac{a}{R}\left(-\dfrac{3}{4}\log\Delta^{-1} + 0.8179\right)$	$\dfrac{1}{3} + \dfrac{1}{4}\Delta^{-1} - \dfrac{1}{8}\dfrac{a}{R}$

TABLE 1. Asymptotic behaviour of the resistance coefficients for small and large separations for small particles ($a \lesssim h \ll R \Longleftrightarrow |\beta| \ll 1$), with subscripts I and II indicating, respectively, no-slip and stress-free obstacle boundaries, e.g. equation (2.2). The $\Delta \ll 1$ asymptotes are obtained applying the method of Cox & Brenner (1967) to the expressions (C 3). The $O(\Delta^0)$ constants in the $\Delta \ll 1$ expressions are computed by a numerical integration of known analytic functions involving polynomials and hyperbolic functions. The asymptotes for $R/a \gg \Delta \gg 1$ are calculated by a regular perturbation expansion of (C 3) for large α; curvature corrections of relative order $h/R \ll 1$ have been retained in this limit.
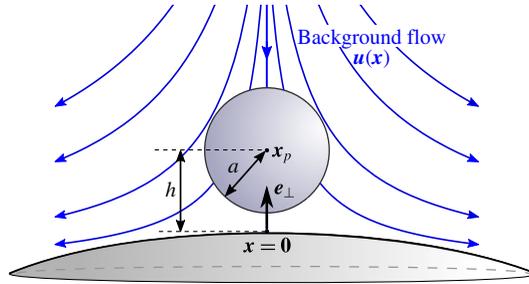
FIGURE 4. (Colour online) A particle held stationary in a normally incident stagnation flow $\boldsymbol{u}(\boldsymbol{x})$ near an obstacle. In the limit of small separations $\Delta \equiv (h/a) - 1 \ll 1$, $\boldsymbol{u}(\boldsymbol{x})$ can be approximated using a Taylor expansion about the point $\boldsymbol{x} = \boldsymbol{0}$ and has a generic structure that depends critically on the boundary conditions at S_w. As a result, the external force required to keep the particle stationary close to the obstacle surface is much greater for a stress-free S_w in comparison with that for a no-slip S_w due to the generally larger surface velocities in the former, see (5.6), (5.7). In both cases, the force on the stationary particle becomes independent of Δ for $\Delta \ll 1$.

as $\boldsymbol{u}(\boldsymbol{x}) \sim \boldsymbol{u}|_{\boldsymbol{x}=\boldsymbol{0}} + \boldsymbol{x} \cdot (\boldsymbol{\nabla}\boldsymbol{u})_{\boldsymbol{x}=\boldsymbol{0}} + (1/2)\boldsymbol{x}\boldsymbol{x} : (\boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u})_{\boldsymbol{x}=\boldsymbol{0}} + \cdots$, cf. figure 4. To compute the normal component of the force, one need consider only the normal velocity $\boldsymbol{u} \cdot \boldsymbol{e}_\perp$ due to the form of (4.12). Evaluating moments of this velocity field at the particle centre $\boldsymbol{x}_p = h\boldsymbol{e}_\perp$ and applying (4.12) results in an expression for the normal force $F_\perp$.

For a no-slip surface S_w, both the normal velocity and the extension rate at the surface of the obstacle are zero. In this case, we find that force contributions proportional to Δ^{-1} and $\log \Delta^{-1}$ both vanish, so that to leading order in a/R, we are left with

$$F_{\perp,\mathrm{I}} \sim 3.2294 \times 6\pi\mu a^3 \times \tfrac{1}{2}\boldsymbol{e}_\perp\boldsymbol{e}_\perp\boldsymbol{e}_\perp \vdots \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u}\big|_{\boldsymbol{x}=\boldsymbol{0}}, \quad \text{for } \Delta \ll 1, \qquad (5.6)$$

in agreement with the results of Goren & O'Neill (1971) for a planar no-slip surface. If the obstacle has a stress-free surface (e.g. a gas bubble), the above result changes qualitatively due to the generally non-zero extension rates at its surface. Using the resistances in table 1, we now find (to leading order in a/R)

$$F_{\perp,\mathrm{II}} \sim 6\pi\mu a \left\{ 1.7887\, a\, \boldsymbol{e}_\perp\boldsymbol{e}_\perp : \boldsymbol{\nabla}\boldsymbol{u} + 0.9021\, a^2\, \boldsymbol{e}_\perp\boldsymbol{e}_\perp\boldsymbol{e}_\perp \vdots \boldsymbol{\nabla}\boldsymbol{\nabla}\boldsymbol{u} \right\}_{\boldsymbol{x}=\boldsymbol{0}} \quad \text{for } \Delta \ll 1. \ (5.7)$$

Thus, for a stress-free S_w, the normal force scales with a^2 at leading order and depends on the extension rate of the flow. We note that for both no-slip and stress-free surfaces, the normal force is independent of Δ for $\Delta \ll 1$. We remark that explicit dependence of the force on the curvature of the obstacle appears only as $O(a/R)$ corrections to the numerical factors in (5.6) and (5.7), although we recognize that the background flow $\boldsymbol{u}(\boldsymbol{x})$ carries an implicit dependence on the curvature of the boundary S_w.

Since objects in Stokes flows are free of net force, the external force required to hold the particle in place has a normal component $F_\perp^{ext} = -F_\perp$. From (5.6) and (5.7), it is evident that the normal component of the force required to keep a small particle ($a/\ell \ll 1$, $a/R \ll 1$) stationary near a stress-free surface is an $O(\ell/a)$ greater than the corresponding force for a particle near a no-slip boundary. While the origin of this feature is evident from the vanishing value of the extension rate $\boldsymbol{e}_\perp\boldsymbol{e}_\perp : \boldsymbol{\nabla}\boldsymbol{u}$ near a

no-slip surface, it is a somewhat counterintuitive result given that the hydrodynamic force experienced by a particle translating normal to a surface at a given velocity and separation distance is smaller for a stress-free surface than for a no-slip surface (by a factor of $1/4$ for $\Delta \ll 1$, cf. table 1). The qualitative difference between forces on stationary particles near no-slip and stress-free surfaces has implications for the flow-driven trapping of particles at interfaces.

5.3. *Motion of a force-free particle*

To analyse the motion of a particle moving under zero external force, it is convenient to scale the background velocity by u_0 and lengths by ℓ and introduce $\boldsymbol{U} = \boldsymbol{u}/u_0$ and $\boldsymbol{X} = \boldsymbol{x}/\ell$. Denoting the dimensionless position of a particle by $\boldsymbol{X}_p \equiv \boldsymbol{x}_p/\ell$ and neglecting both particle and fluid inertia, the trajectory of a particle with no external force is given as

$$\frac{\mathrm{d}\boldsymbol{X}_p(t)}{\mathrm{d}t} = \boldsymbol{U}(\boldsymbol{X}_p(t)) + \boldsymbol{W}\left(\boldsymbol{X}_p(t), \frac{a}{\ell}\right), \quad (5.8)$$

where t is a dimensionless time (in units of ℓ/u_0). Here, $\boldsymbol{W}$ is the velocity of the sphere relative to the mean fluid velocity in its vicinity, which becomes zero when $a/\ell = 0$. From the expression (4.12), the normal component of the relative velocity $\boldsymbol{W}$ of a force-free particle can be written as

$$W_\perp \equiv \boldsymbol{e}_\perp \cdot \boldsymbol{W} = \boldsymbol{e}_\perp \cdot \left\{ -\frac{a}{\ell}\frac{\mathcal{B}}{\mathcal{A}}\boldsymbol{e}_\perp \cdot \tilde{\boldsymbol{\nabla}}\boldsymbol{U} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2 \frac{\mathcal{C}}{\mathcal{A}}\boldsymbol{e}_\perp\boldsymbol{e}_\perp : \tilde{\boldsymbol{\nabla}}\tilde{\boldsymbol{\nabla}}\boldsymbol{U} + \frac{1}{2}\left(\frac{a}{\ell}\right)^2 \frac{\mathcal{D}}{\mathcal{A}}\tilde{\nabla}^2\boldsymbol{U} \right\}$$
$$+ O\left(\left|\frac{a^3}{\ell^3}\tilde{\boldsymbol{\nabla}}\tilde{\boldsymbol{\nabla}}\tilde{\boldsymbol{\nabla}}\boldsymbol{U}\right|\right), \quad (5.9)$$

where $\tilde{\boldsymbol{\nabla}} = \ell\boldsymbol{\nabla}$ is a dimensionless gradient operator.

The ratios of the resistance coefficients in (5.9) for particles near rigid and stress-free boundaries are reported in figure 5. As $\Delta \to \infty$, both $\mathcal{B}/\mathcal{A}$ and $\mathcal{C}/\mathcal{A}$ decay to zero, while $\mathcal{D}/\mathcal{A} \to 1/3$, resulting in $\boldsymbol{W} \sim (1/6)(a/\ell)^2\tilde{\nabla}^2\boldsymbol{U}$ as per Faxén's law (quadratic in a/ℓ). However, the effect of the extensional gradient term, although decaying with Δ, can be significant due to its linear dependence on a/ℓ. The contribution of background flow gradients to the relative velocity is important as long as

$$\frac{\ell}{a}\frac{2B}{\mathcal{D}} \gtrsim O(1) \quad \text{and} \quad \frac{\ell}{a}\frac{2B}{\mathcal{C}} \gtrsim O(1). \quad (5.10a,b)$$

For small separations ($\Delta \ll 1$), it follows from the asymptotic forms of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ that

$$\frac{\ell}{a}\frac{2\mathcal{B}}{\mathcal{D}} \sim \left(\frac{\ell}{a}\right)\frac{2}{\Delta \log \Delta^{-1}} \gg 1 \quad \text{and} \quad \frac{\ell}{a}\frac{2\mathcal{B}}{\mathcal{C}} \sim \frac{2\ell}{a} \gg 1. \quad (5.11a,b)$$

Thus, from (5.9), the extensional gradient contribution to the relative velocity is dominant over the curvature contributions at small separations. Also, we note that the contribution from curvature terms proportional to $\mathcal{C}$ are much greater than those proportional to $\mathcal{D}$ for $\Delta \lesssim 1$.

In the limit $\Delta \gg 1$, we find $\mathcal{B} \propto \Delta^{-2}$, $\mathcal{C} \propto \Delta^{-3}$ and $\mathcal{D} \sim 1/3$ with $O(1)$ prefactors for $\mathcal{B}$ and $\mathcal{C}$, both for the no-slip and no-stress cases (cf. table 1). For large separations,
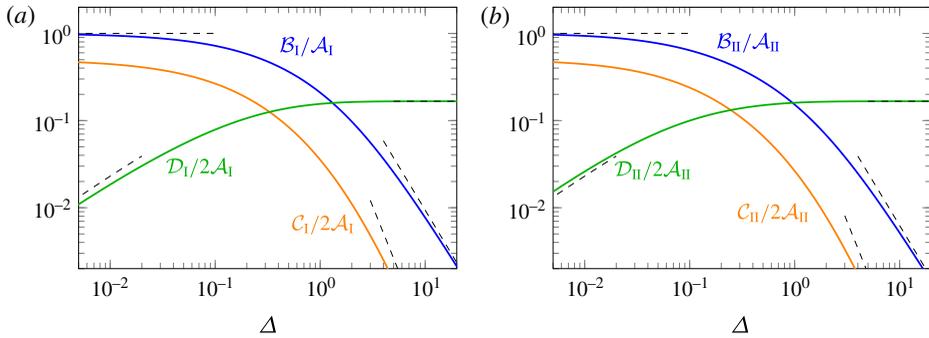
FIGURE 5. (Colour online) Ratios $\mathcal{B}/\mathcal{A}$, $\mathcal{C}/2\mathcal{A}$ and $\mathcal{D}/2\mathcal{A}$ that govern the motion of a force-free particle relative to the fluid for (*a*) no-slip and (*b*) stress-free conditions on the obstacle in the limit $a/R \to 0$. Note that the velocity of the particle relative to the fluid due to flow curvature (coefficients $\mathcal{C}/2\mathcal{A}$ and $\mathcal{D}/2\mathcal{A}$) is multiplied by an $O(a^2/\ell^2)$ factor, while the contribution of extensional flow gradients (quantified by $\mathcal{B}/2\mathcal{A}$) is multiplied by a much larger $O(a/\ell)$ factor. The relative importance of the terms therefore depends both on Δ and a/ℓ in addition to the flow properties. Dashed lines are leading-order asymptotes for large and small Δ and follow from the results in table 1.

the effect of the extensional gradient term is therefore important for distances of the particle to the boundary up to

$$\Delta \lesssim \sqrt{\frac{\ell}{a}} \quad \text{or} \quad h \lesssim \sqrt{a\ell}. \tag{5.12a,b}$$

Thus, the classical Faxén's law for spheres only applies when $h \gg \sqrt{a\ell} \gg a$, and boundary effects become important even for modest distances from no-slip or stress-free boundaries. The contribution of extensional flow gradients to the relative velocity of the particle to the fluid becomes increasingly important with decreasing h and is ultimately dominant when the separation between the two surfaces is comparable to the particle size, i.e. when $\Delta = O(1)$.

The above conclusions remain valid when corrections for the finite curvature of the boundary are taken into account, provided that $a \lesssim h \ll R$. In particular, since the motion of the particle relative to the fluid depends on ratios of the dimensionless hydrodynamic resistances, the resulting relative velocity $W_\perp$ in (5.9) is given to leading order by its $a/R \to 0$ limit (planar surface S_w) and has corrections of $O(a/R)$, cf. table 1.

By calculating tangential components of the force in terms of moments of the background flow, expressions similar to (5.9) can be developed for the tangential components of the relative velocity of a force-free particle. It is noted that the presence of boundaries in general couples rotation with wall-parallel translation, so that an additional condition on the external torque acting on the particle must be specified to uniquely determine its motion (e.g. zero external torque acts on particles in several common applications). These mobility relations are known to varying degree of accuracy in linear flows (Goldman *et al.* 1967*b*; Jeffrey 1992; Swan & Brady 2007; Aponte-Rivera & Zia 2016) and in the far field ($h \gg a$) for quadratic flows (Haber & Brenner 1999). The present results can be integrated with well-developed numerical techniques for particle dynamics (Brady & Bossis 1988; Sierou & Brady 2001; Swan & Brady 2011) to accurately compute the trajectories of particles in many-body systems.

6. Conclusions

We have determined the hydrodynamic force, normal to a large obstacle, on a nearby spherical particle immersed in a background flow. By a local expansion of the background flow up to quadratic order, we obtained analytical expressions for the normal force in terms of moments of the background flow in the vicinity of the particle. Our results, which hold for spheres both near rigid walls as well as near free surfaces in the small capillary number limit, provide a first step towards a systematic generalization of Faxén's law in confined systems.

We identify two important modifications that relate to Faxén-like contributions when a spherical particle is near a boundary. First, there is a contribution to the force proportional to extensional flow gradients, which forms the dominant contribution to the relative velocity between the particle and the fluid when the separation between the particle and the obstacle is comparable to the particle size. Second, we show that in addition to the Faxén force term, an extra force contribution proportional to curvature of the flow normal to the obstacle becomes important when the distance to the wall is comparable to the particle size. The near-field force contributions automatically recover the results of lubrication theory and can be integrated into dynamical simulations of particle trajectories near obstacles. We apply our theory to show that the force required to hold a particle stationary in the vicinity of an obstacle is generally much greater if the obstacle surface is stress free than if the obstacle has a no-slip surface.

Analytical expressions for the normal force have been obtained for arbitrary distances of the particle to the obstacle and the asymptotic behaviour of the force for large and small separations between the particle and the obstacle have been calculated. Both the exact and asymptotic results are useful in describing the transport of particles at low Reynolds numbers in confined geometries, such as in microfluidic devices or in biological systems. The generality of the results obtained here allow them to be used in simulations of particles near physical boundaries without the need to resolve the flow on the scale of individual particles and the thin lubrication layers between surfaces in near contact.

Acknowledgements

Appendix A. Coefficients a_n, b_n, c_n and d_n in the model flow problem

The model problem consists of the viscous flow driven by the translation of a sphere of radius a normal to a fixed spherical obstacle of radius R in an otherwise unbounded fluid that is quiescent at infinity. The flow is axisymmetric and can thus be expressed in terms of a dimensionless streamfunction $\psi'(\tilde{r}, \tilde{z})$ (in units of $V'a^2$) that is related to dimensionless velocity components (in units of V') by $v'_r = \tilde{r}^{-1} \partial \psi' / \partial \tilde{z}$ and $v'_z = -\tilde{r}^{-1} \partial \psi' / \partial \tilde{r}$. In bipolar coordinates (ξ, η), the streamfunction ψ' in the model problem can be generally expressed as $\psi' = \sum_{n=1}^{\infty} U_n(\xi) V_n(\tau)$, with U_n and $V_n(\tau)$ are defined in (4.3). The coefficients a_n, b_n, c_n and d_n in U_n are determined by the boundary conditions at the two spherical surfaces of the particle S_p and the obstacle S_w and depend on α and β, which are related to the dimensionless separation distance h/a and the curvature ratio a/R, cf. (4.1).

For the case where S_w is a no-slip surface, the coefficients a_n, b_n, c_n and d_n can be reconstructed as a superposition of the results of Maude (1961) (summarized in Adamczyk *et al.* 1983) for co-moving and counter-moving solid spheres normal to their line of centres. Defining

$$\mathcal{X} = \alpha + \beta, \quad \mathcal{Y} = \alpha - \beta \quad \text{and}$$
$$\kappa_{\mathrm{I}} = \frac{n(n+1)\sinh^2 \alpha}{\sqrt{2}(2n-1)(2n+3)\left\{ 4\sinh^2\left(n+\frac{1}{2}\right)\mathcal{Y} - (2n+1)^2 \sinh^2 \mathcal{Y} \right\}}, \tag{A 1}$$

the coefficients are

$$
\begin{aligned}
a_{n,\mathrm{I}} = {} & \frac{\kappa_{\mathrm{I}}}{2}(2n+3)\Bigg\{ (2n+1)^2 e^{\mathcal{Y}}\sinh \mathcal{Y} - (2n+1)(2n-1)e^{-\mathcal{X}}\sinh \mathcal{Y} \\
& - 2(2n+1)e^{-(n-1/2)\mathcal{X}}\sinh\left(n+\frac{3}{2}\right)\mathcal{Y} + 2(2n-1)e^{-(n+1/2)\mathcal{X}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \\
& + 4e^{-(n+1/2)\mathcal{Y}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \Bigg\},
\end{aligned} \tag{A 2a}
$$

$$
\begin{aligned}
b_{n,\mathrm{I}} = {} & -\frac{\kappa_{\mathrm{I}}}{2}(2n+3)\Bigg\{ (2n+1)^2 e^{\mathcal{Y}}\sinh \mathcal{Y} - (2n+1)(2n-1)e^{-\mathcal{X}}\sinh \mathcal{Y} \\
& + 2(2n+1)e^{-(n-1/2)\mathcal{X}}\sinh\left(n+\frac{3}{2}\right)\mathcal{Y} - 2(2n-1)e^{-(n+1/2)\mathcal{X}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \\
& + 4e^{-(n+1/2)\mathcal{Y}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \Bigg\},
\end{aligned} \tag{A 2b}
$$

$$
\begin{aligned}
c_{n,\mathrm{I}} = {} & \frac{\kappa_{\mathrm{I}}}{2}(2n-1)\Bigg\{ (2n+1)^2 e^{-\mathcal{Y}}\sinh \mathcal{Y} - (2n+1)(2n+3)e^{\mathcal{X}}\sinh \mathcal{Y} \\
& - 2(2n+1)e^{-(n+3/2)\mathcal{X}}\sinh\left(n-\frac{1}{2}\right)\mathcal{Y} + 2(2n+3)e^{-(n+1/2)\mathcal{X}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \\
& - 4e^{-(n+1/2)\mathcal{Y}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \Bigg\} \quad \text{and}
\end{aligned} \tag{A 2c}
$$

$$
\begin{aligned}
d_{n,\mathrm{I}} = {} & -\frac{\kappa_{\mathrm{I}}}{2}(2n-1)\Bigg\{ (2n+1)^2 e^{-\mathcal{Y}}\sinh \mathcal{Y} - (2n+1)(2n+3)e^{\mathcal{X}}\sinh \mathcal{Y} \\
& + 2(2n+1)e^{-(n+3/2)\mathcal{X}}\sinh\left(n-\frac{1}{2}\right)\mathcal{Y} - 2(2n+3)e^{-(n+1/2)\mathcal{X}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \\
& - 4e^{-(n+1/2)\mathcal{Y}}\sinh\left(n+\frac{1}{2}\right)\mathcal{Y} \Bigg\},
\end{aligned} \tag{A 2d}
$$

valid for arbitrary α and β. The subscript I denotes association with BC I, i.e. both surfaces are no slip. The above coefficients reduce to the case of a sphere near a no-slip planar wall (Brenner 1961) when $\beta = 0$ and can be written as

$$
a_{n,\mathrm{I}} = -c_{n,\mathrm{I}} = \frac{n(n+1)(2n+1)\sinh^4 \alpha}{\sqrt{2}\left[4\sinh^2\left(n+\frac{1}{2}\right)\alpha - (2n+1)^2 \sinh^2 \alpha \right]}, \tag{A 3a}
$$

$$b_{n,\mathrm{I}} = -\frac{2n+3}{2n-1}d_{n,\mathrm{I}}$$

$$= \frac{n(n+1)\sinh^2\alpha}{\sqrt{2}(2n-1)}\left[1 - \frac{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha}{4\sinh^2\left(n+\frac{1}{2}\right)\alpha - (2n+1)^2\sinh^2\alpha}\right]. \quad (A\,3b)$$

Two-sphere relations similar to (A 2) can also be obtained for the hydrodynamic interaction of a no-slip spherical particle S_p at $\xi = \alpha$ with a stationary stress-free spherical surface S_w at $\xi = \beta$ by the appropriate choice of the coefficients a_n, b_n, c_n and d_n. These coefficients are calculated here using an approach similar to that of Brenner (1961); defining

$$\kappa_{\mathrm{II}} = \frac{n(n+1)\sinh^2\alpha}{\sqrt{2}(2n-1)(2n+3)\{2\sinh(2n+1)\mathcal{Y} - (2n+1)\sinh 2\mathcal{Y}\}}, \quad (A\,4)$$

we find

$$b_{n,\mathrm{II}} = -2\kappa_{\mathrm{II}}(2n+3)\,\mathrm{e}^{-(n+1/2)\alpha}\left\{2\mathrm{e}^\alpha\cosh\left(n+\tfrac{3}{2}\right)\mathcal{Y}\right.$$
$$\left. + (2n-1)\mathrm{e}^{(n+3/2)\mathcal{Y}}\sinh\alpha\right\}\cosh\left(n-\tfrac{1}{2}\right)\beta, \quad (A\,5)$$

$$d_{n,\mathrm{II}} = 2\kappa_{\mathrm{II}}(2n-1)\,\mathrm{e}^{-(n+1/2)\alpha}\left\{2\mathrm{e}^{-\alpha}\cosh\left(n-\tfrac{1}{2}\right)\mathcal{Y}\right.$$
$$\left. + (2n+3)\mathrm{e}^{(n-1/2)\mathcal{Y}}\sinh\alpha\right\}\cosh\left(n+\tfrac{3}{2}\right)\beta, \quad (A\,6)$$

$$a_{n,\mathrm{II}} = -b_{n,\mathrm{II}}\tanh\left(n-\tfrac{1}{2}\right)\beta \quad\text{and} \quad (A\,7)$$

$$c_{n,\mathrm{II}} = -d_{n,\mathrm{II}}\tanh\left(n+\tfrac{3}{2}\right)\beta, \quad (A\,8)$$

where the subscript II denotes an association with BC II (stress-free obstacle). The results of Brenner (1961) for a sphere translating normal to a planar stress-free surface can be recovered by setting $\beta = 0$ in the above expressions; in this case the coefficients reduce to

$$a_{n,\mathrm{II}} = c_{n,\mathrm{II}} = 0, \quad (A\,9a)$$

$$b_{n,\mathrm{II}} = -\frac{n(n+1)\sinh^2\alpha}{\sqrt{2}(2n-1)}\left[\frac{4\cosh^2\left(n+\frac{1}{2}\right)\alpha + 2(2n+1)\sinh^2\alpha}{2n\sinh(2n+1)\alpha - (2n+1)\sinh 2\alpha} - 1\right], \quad (A\,9b)$$

$$d_{n,\mathrm{II}} = -\frac{n(n+1)\sinh^2\alpha}{\sqrt{2}(2n+3)}\left[1 - \frac{4\cosh^2\left(n+\frac{1}{2}\right)\alpha - 2(2n+1)\sinh^2\alpha}{2n\sinh(2n+1)\alpha - (2n+1)\sinh 2\alpha}\right]. \quad (A\,9c)$$

Appendix B. Evaluating the integrals for the force

The expression for the force in (4.8) relies on computing the integrals $\mathcal{I}_k$ and $\mathcal{J}$ defined in (4.9), which we give here again:

$$\mathcal{I}_k \equiv \frac{1}{6}\int_{-1}^{1}\frac{(\sinh\xi)^{3k-1}}{(\cosh\xi - \tau)^{2k+3}}\frac{\partial}{\partial\xi}\left\{\left(\frac{\cosh\xi - \tau}{\sinh\xi}\right)^k g(\xi,\tau)\right\}\,\mathrm{d}\tau \quad\text{and} \quad (B\,1a)$$

$$\mathcal{J} \equiv \frac{1}{24} \int_{-1}^{1} \frac{(1-\tau^2) \sinh \xi}{(\cosh \xi - \tau)^7} \frac{\partial}{\partial \xi} \{(\cosh \xi - \tau)^2 g(\xi, \tau)\} \, d\tau, \qquad (B\,1b)$$

where $g(\xi, \tau)$ is defined in (4.5). Here, we outline the method used to evaluate these integrals.

Both $\mathcal{I}_k$ and $\mathcal{J}$ involve integration with respect to τ (which parametrizes the surface of the sphere), and have kernels containing $V_n(\tau)$, which enter through $g(\xi, \tau)$. We recall the definition of $V_n(\tau)$ in terms of Legendre polynomials $P_n(\tau)$ and give some properties (Stimson & Jeffery 1926) that are useful in the evaluation of the integrals:

$$V_n(\tau) \equiv \frac{P_{n-1}(\tau) - P_{n+1}(\tau)}{2n+1}, \qquad (B\,2a)$$

$$(1-\tau^2)\frac{d^2 V_n}{d\tau^2} + n(n+1)V_n = 0, \qquad (B\,2b)$$

$$\frac{dV_n}{d\tau} = -P_n(\tau), \qquad (B\,2c)$$

$$(1-\tau^2)P_n(\tau) = \frac{(n+1)(n+2)}{2n+1}V_{n+1} - \frac{n(n-1)}{2n+1}V_{n-1}. \qquad (B\,2d)$$

Rewriting derivatives of $V_n(\tau)$ in $g(\xi, \tau)$ using (B 2), we obtain

$$
\begin{aligned}
g(\xi, \tau) \;=\; & (\cosh \xi - \tau)^{5/2} \sum_{n=0}^{\infty} \left[V_n(\tau) \left\{ \frac{d^2 U_n(\xi)}{d\xi^2} - \frac{2 \sinh \xi}{\cosh \xi - \tau} \frac{dU_n(\xi)}{d\xi} \right. \right. \\
& \left. + \left(\frac{3 \cosh \xi}{\cosh \xi - \tau} - \frac{9}{4} - n(n+1) \right) U_n(\xi) \right\} \\
& \left. + \frac{2 U_n(\xi)}{\cosh \xi - \tau} \left\{ \frac{n(n-1)}{2n+1} V_{n-1}(\tau) - \frac{(n+1)(n+2)}{2n+1} V_{n+1}(\tau) \right\} \right], \quad (B\,3)
\end{aligned}
$$

where $U_n(\xi)$ is defined in (4.3a). We observe that $g(\xi, \tau)$ is a linear combination of $V_n(\tau)$ for different n with coefficients that depend only on ξ. Note also that $(1-\tau^2)V_n$, which appears in the kernel of $\mathcal{J}$, can be written in terms of V_{n-2}, V_n and V_{n+1} using (B 2).

Any additional τ dependence (i.e. outside of $V_n(\tau)$) in $g(\xi, \tau)$, and therefore in the kernels of $\mathcal{I}_k$ and $\mathcal{J}$, is entirely due to factors of $(\cosh \xi - \tau)^{m/2}$, where m is an odd integer. Both $\mathcal{I}_k$ and $\mathcal{J}$ can therefore be written as superposition of integrals $\mathcal{K}_{m,n}(\xi)$ defined by

$$\mathcal{K}_{m,n}(\xi) \equiv \int_{-1}^{1} \frac{V_n(\tau)}{(\cosh \xi - \tau)^{m/2}} \, d\tau, \quad \text{where } m \text{ odd.} \qquad (B\,4)$$

The coefficients of the superposition, which are obtained by substituting (B 3) into (B 1) and rearranging, are functions of ξ only, and remain constant throughout the integration. Taking a ξ derivative of (B 4) immediately results in an expression for $\mathcal{K}_{m+2,n}(\xi)$ in terms of $\mathcal{K}_{m,n}(\xi)$, thus providing a recursion relation. Applying the formula for $\mathcal{K}_{1,n}(\xi)$ given by Stimson & Jeffery (1926), we obtain

$$\mathcal{K}_{m,n}(\xi) = -\frac{2}{(m-2) \sinh \xi} \frac{d\mathcal{K}_{m-2,n}(\xi)}{d\xi}, \qquad (B\,5)$$

with

$$\mathcal{K}_{1,n}(\xi) = \frac{2\sqrt{2}}{2n+1} \left\{ \frac{e^{-(n-1/2)\xi}}{2n-1} - \frac{e^{-(n+3/2)\xi}}{2n+3} \right\}, \tag{B 6}$$

which can be used to compute $\mathcal{K}_{m,n}(\xi)$ for any odd m. Substituting these expressions into $\mathcal{I}_k$ and $\mathcal{J}$ results, after evaluation at $\xi = \alpha$, in formulas only involving α and the coefficients $a_n(\alpha, \beta)$, $b_n(\alpha, \beta)$, $c_n(\alpha, \beta)$ and $d_n(\alpha, \beta)$, which enter through $U_n(\xi)$. The expressions for the force (4.10) and resistance coefficients (4.11) in terms of a_n, b_n, c_n and d_n then follow directly.

Appendix C. Expressions for the hydrodynamic resistances for a spherical particle near a planar obstacle ($\beta = 0$)

Analytical expressions for the dimensionless hydrodynamic resistances $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ can be obtained by substituting the expressions for a_n, b_n, c_n and d_n from either (A 3) or (A 9) as appropriate, into (4.11). This yields infinite series representations for $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ in terms of $\alpha = \cosh^{-1}(h/a)$ only, for both no-slip and no-stress cases, which we provide here. We first define

$$
\mathcal{L}_n(\alpha) \equiv -\frac{\sqrt{2}}{3 \sinh \alpha} \{a_n + b_n + c_n + d_n\}
$$

$$
= \begin{cases}
\dfrac{4 \sinh \alpha}{3} \dfrac{n(n+1)}{(2n-1)(2n+3)} \left[\dfrac{2 \sinh(2n+1)\alpha + (2n+1) \sinh 2\alpha}{4 \sinh^2 \left(n + \frac{1}{2} \right) \alpha - (2n+1)^2 \sinh^2 \alpha} - 1 \right], \\
\qquad \text{for BC I} \\[6pt]
\dfrac{4 \sinh \alpha}{3} \dfrac{n(n+1)}{(2n-1)(2n+3)} \left[\dfrac{4 \cosh^2 \left(n + \frac{1}{2} \right) \alpha + (2n+1)^2 \sinh^2 \alpha}{2 \sinh(2n+1)\alpha - (2n+1) \sinh 2\alpha} - 1 \right], \\
\qquad \text{for BC II,}
\end{cases} \tag{C 1}
$$

and

$$
\mathcal{M}_n(\alpha) \equiv \frac{\sqrt{2}}{3} \left\{ (2n-1)(a_n + b_n) + (2n+3)(c_n + d_n) \right\}
$$

$$
= \begin{cases}
-\dfrac{4 \sinh^4 \alpha}{3} \dfrac{n(n+1)(2n+1)}{4 \sinh^2 \left(n + \frac{1}{2} \right) \alpha - (2n+1)^2 \sinh^2 \alpha}, & \text{for BC I} \\[6pt]
-\dfrac{4 \sinh^4 \alpha}{3} \dfrac{n(n+1)(2n+1)}{2 \sinh(2n+1)\alpha - (2n+1) \sinh 2\alpha}, & \text{for BC II,}
\end{cases} \tag{C 2}
$$

where BC I and BC II represent, respectively, no-slip or stress-free conditions on the surface S_w of the obstacle.

The dimensionless hydrodynamic resistances $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are then

$$
\mathcal{A}(\alpha) = \sum_{n=0}^{\infty} \mathcal{L}_n, \tag{C 3a}
$$

$$\mathcal{B}(\alpha) = \mathcal{A}\cosh\alpha + \sum_{n=0}^{\infty} \mathcal{M}_n, \tag{C 3b}$$

$$\mathcal{C}(\alpha) = 2\mathcal{B}\cosh\alpha - \mathcal{A}\cosh^2\alpha$$
$$- \sum_{n=0}^{\infty}\left\{(n^2+n-1)\mathcal{L}_n \sinh^2\alpha + \frac{2}{3}(2n+1)\mathcal{M}_n\sinh\alpha\right\}, \tag{C 3c}$$

$$\mathcal{D}(\alpha) = \frac{\mathcal{A}}{3} - \frac{\mathcal{C}}{5} + \frac{2\sinh\alpha}{15}\sum_{n=0}^{\infty}e^{-(2n+1)\alpha}\left\{\left(4(n^2+n-1)\sinh\alpha\right.\right.$$
$$\left.\left. - (2n+1+\coth\alpha)\cosh\alpha\right)\mathcal{L}_n \pm (2n+1-3\coth\alpha)\mathcal{M}_n\right\}, \tag{C 3d}$$

with $\mathcal{L}_n$, $\mathcal{M}_n$ chosen appropriate to the boundary condition at S_w and the positive and negative signs in front of the last term in (C 3d) correspond to stress-free and no-slip conditions, respectively.

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